

COLLISIONS OF SINGULARITIES IN ELLIPTIC FIBRATIONS

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1. INTRODUCTION

Definition 1.1. An *Elliptic Curve* is a smooth projective curve of genus 1 with a choice of a rational point.

Example 1.1. The *Fermat cubic* defined as the vanishing $V(x_0^3 + x_1^3 + x_2^3)$ in $\mathbf{P}^2$.

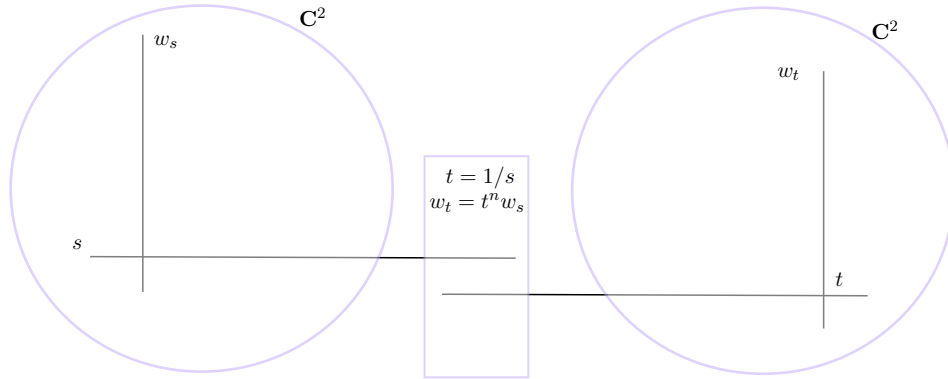
In fact, we will show that we may take any smooth cubic in $\mathbf{P}^2$, i.e. a cubic with empty singular locus. Where the singular locus of some variety defined by F is the vanishing $V(\partial_0 F, \partial_1 F, \partial_2 F, F)$.

Example 1.2. Letting $F = x_0^3 - x_1 x_2^2$, we find that the singular locus of F is defined by the vanishing $V(3x_0^2, -x_2^2, -2x_1 x_2, F)$. This implies that $x_0 = x_2 = 0$, and so the singular locus contains the point $[0 : 1 : 0]$.

It turns out that complete intersection of two quadrics in $\mathbf{P}^3$ will also product elliptic curves. To see this we will make use of the adjunction formula.

1.1. Line Bundles on $\mathbf{P}^1$. We begin by defining the line bundles $\mathcal{O}(n)$ over $\mathbf{P}^1$ to start building up some intuition on these objects. We let $[x_0 : x_1]$ denote our coordinates on $\mathbf{P}^1$, that is, $\mathbf{P}^1 = \text{Proj } \mathbf{C}[x_0, x_1]$. We note that $\mathbf{P}^1$ is covered by two open sets $D(x_i) \cong \mathbf{C}$, where $x_i \neq 0$. We let $s = x_1/x_0$ denote the affine coordinate on $D(x_0)$ and $t = x_0/x_1$ denote the coordinate on $D(x_1)$.

Now the data of a line bundle on $\mathbf{P}^1$ consists of a $\mathbf{C}$ as the fiber over every point in $\mathbf{P}^1$ along with a gluing data. We prescribe this data over our two open sets, then glue along their intersection using a transition function. Let w_s denote the coordinate on each fiber of the s -line, and similarly w_t the coordinate on fibers over the t -line. We need to define what happens when we move from one open chart to the next.



Definition 1.2. We define the line bundle $\mathcal{O}(n)$ by the transition data
$$\begin{cases} t = 1/s \\ w_t = t^n w_s. \end{cases}$$

1.2. Adjunction in $\mathbf{P}^n$. Let H be the class of a hyperplane in $\mathbf{P}^n$. We then define the total Chern class of $\mathbf{P}^n$ to be $c(\mathbf{P}^n) = (1 + H)^{n+1}$. Now let X be a complete intersection $V(d_1, d_2, \dots, d_k) = X$. Where the d_i denote the degree of the defining equation for the hyper surface.

Definition 1.3. The *total Chern class* of X is

$$c(X) := \frac{c(\mathbf{P}^n)}{(1 + d_1 H) \cdots (1 + d_k H)}.$$

Note that since X is a complete intersection $\dim X = n - k$. We may expand out this formula with respect to H to define the Chern classes $c_i(X)$,

$$c(X) = 1 + c_1(X)H + c_2(X)H^2 + \cdots + c_{n-k}(X)H^{n-k}.$$

In particular for us, we find that $c_1(X) = n + 1 - \sum_i d_i$. For now we will assume without proof that genus = 1 if and only if $c_1(X) = 0$.

Using the first Chern class, we will analyze what can happen in $\mathbf{P}^2$ and $\mathbf{P}^3$.

- In $\mathbf{P}^2$, we are looking for a curve $X = V(d_1)$, that is the vanishing of a single degree d_1 equation. By the adjunction formula $c_1(X) = 0$ if and only if $3 - d_1 = 0$. So X is a cubic.
- In $\mathbf{P}^3$, we must cut down our dimension by 2 to get a curve, so $X = V(d_1, d_2)$. Again, $c_1(X) = 0$ if and only if $d_1 + d_2 = 4$. There are two possible cases $d_1 = 1, d_2 = 3$. Then we are considering the complete intersection of a hyperplane and a cubic surface, but this is redundant as $\mathbf{P}^3$ cut by a hyperplane is just $\mathbf{P}^2$. The second possibility is $d_1 = d_2 = 2$, which is the complete intersection of two quadric surfaces.

1.3. Weighted Projective Space. Let $w = (w_0, w_1, \dots, w_n) \in \mathbf{Z}_{>0}^{n+1}$, be a weight vector.

Definition 1.4. The *weighted projective space* with weights w is

$$\mathbf{P}^w := \text{Proj } \mathbf{C}[x_0, \dots, x_n],$$

where x_i has degree w_i .

Using coordinates, we may view $\mathbf{P}^w$ as the points in projective space with

$$[x_0 : \cdots : x_n] \sim [\lambda^{w_0} x_0 : \cdots : \lambda^{w_n} x_n],$$

for $\lambda \in \mathbf{C}^\times$.

Example 1.3. If $w = (1, 1, 1, \dots, 1)$ then $\mathbf{P}^w = \mathbf{P}^n$.

Example 1.4. If $w = (1, 2)$, then $\mathbf{P}^w$ consists of points where $[x_0 : x_1] = [\lambda x_0 : \lambda^2 x_1]$. On $D(x_0)$, we may let $\lambda = 1/x_0$, then our points look like $[1 : x_1/x_0^2]$, and we define $s = x_1/x_0^2$ to be the affine coordinate. Now on $D(x_1)$, to make our second coordinate 1, we must let $\lambda^2 = 1/x_1$ giving us two possible choices for λ , and hence our coordinate t on $D(x_1)$. But if we let $t' = t^2$, which amounts to the morphism $(s, t) \mapsto (s, t^2)$ then we find that the transition function becomes $t' = 1/s$. Note that this is the same transition function as that of $\mathbf{P}^1$, so $\mathbf{P}^w \cong \mathbf{P}^1$.

Remark 1.5. More generally we have that $\mathbf{P}^w \cong \mathbf{P}^1$, when $w = (n, m)$.

Now let $w = (1, 1, 2)$, and let the coordinates on $\mathbf{P}^w$ be $[x : z : y] \sim [\lambda x : \lambda z : \lambda^2 y]$. Then $c(\mathbf{P}^w) = (1 + H)(1 + H)(1 + 2H)$, since these are the corresponding degrees of the variables. Most of the Chern class details are being black boxed. Suppose that X is the vanishing locus of a section of $\mathcal{O}_{\mathbf{P}^w}(d)$. Then

$$c(X) = \frac{(1 + H)^2(1 + 2H)}{(1 + dH)} = 1 + (4 - d)H + \cdots$$

So $c_1(X) = 0$ if and only if $d = 4$. Hence a quartic in $\mathbf{P}^w$ defines a genus 1 curve.

Example 1.5. The **Jacobi form** $y^2 + x^4 + z^4 = 0$ defines a genus 1 curve in $\mathbf{P}^w$, where $w = (1, 1, 2)$.

Example 1.6. If $w = (1, 2, 3)$ then a curve of degree 6 defines a genus 1 curve in $\mathbf{P}^w$. Let $[z : x : y]$ denote the coordinates on our weighted projective space. Then the **Weierstrass equation**

$$y^2 = x^3 + fxz^4 + gz^6$$

gives a genus 1 curve, where f is a section of $\mathcal{O}_{\mathbf{P}^w}(4)$, and g is a section of $\mathcal{O}_{\mathbf{P}^w}(6)$. We will revisit this later in the context of projective bundles.

2. FURTHER STUDY OF WEIGHTED PROJECTIVE SPACES

Example 2.1. We will show that $\mathbf{P}^{w_0, w_1} \cong \mathbf{P}^{1,1} = \mathbf{P}^1$. On $D(x_0)$, we can normalize our coordinates to look like $[1 : x_0^{-w_1/w_0} x_1]$. Similarly on $D(x_1)$, we normalize our coordinates to look like $[x_0 x_1^{-w_0/w_1} : 1]$. Consider the morphism $(a, b) \mapsto (s = a^{w_0}, t = b^{w_1})$. Then $s = x_0^{-w_1} x_1^{w_0}$, and $t = x_0^{w_1} x_1^{-w_0}$. So we note that $s = 1/t$ which is the same transition function of $\mathbf{P}^1$.

Example 2.2. Let $w = (2, 2, 1)$, to have an elliptic curve on $\mathbf{P}^w$, by the adjunction formula, it must be a degree 5 hypersurface. Letting $[x_0 : x_1 : x_2]$ be the coordinates on $\mathbf{P}^w$, we write out a general degree 5 polynomial in these variables.

$$A_0 x_2^5 + (Bx_0 + Cx_1)x_2^3 + Q(x_0, x_1)x_2,$$

where $Q(x_0, x_1)$ is a quadratic in the variables. Note then this variety would not be irreducible since we may factor out x_2 . Hence there are no elliptic curves on $\mathbf{P}^w$.

So we shouldn't expect every weighted projective space to have elliptic curves. Here are some cases where they do,

- $\mathbf{P}^{1,1,2}$: in this case this space is called the $\tilde{E}_7$ model
- $\mathbf{P}^{1,2,3}$: $\tilde{E}_8$ model
- $\mathbf{P}^{1,1,1}$: $\tilde{E}_6$ model.

Now, we try to find rational points on our elliptic curves. We consider an elliptic curve in the $\tilde{E}_8$ model, with coordinates $[z : x : y]$. Then $y^2 = x^3 + fxz^2 + gz^3$. We begin by naively setting $z = 0$, then $y^2 = x^3$. Then a choice of point would be $[0 : x : y] \sim [0 : \lambda^2 x : \lambda^3 y]$. If we let $x = 1$, then $\lambda = \pm 1$ so $y^2 = 1$, and so a rational point would be $[0 : 1 : 1] \sim [0 : 1 : -1]$. They are equivalent by $\lambda = -1$ Hence we have found one rational point.

Now in the $\tilde{E}_7$ model, our elliptic curve would like $y^2 + x^4 + ax^2z^2 + bxz^3 + dz^4$ in general. where our coordinates are again $[z : x : y]$. Letting $z = 0$, we have in the affine piece that $y^2 = x^4$. If we let $x = 1$, then $y^2 = 1$, giving us two separate rational points in this case, namely $[0 : 1 : 1] \neq [0 : 1 : -1]$. Hence in this case we've found two generic rational points. Hence we see that when we transform the Jacobi quartic to the Weierstrass model, something must occur on the level of the rational points.

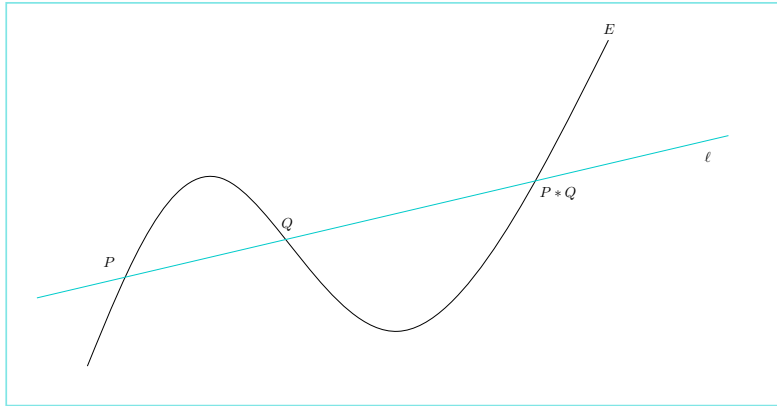
Now in the $\tilde{E}_6$ model we write our general cubic as

$$x^3 + y^3 + \sum_{1 \leq i \leq 3} \sum_{0 \leq j \leq 2} a_{ijk} x^j y^{3-j-i} z^i.$$

Then if we let $z = 0$, with $xy \neq 0$, our affine curve becomes $x^3 = y^3$, so letting $\lambda = 1/x$ we find that our equation simplifies to $y^3 = 1$. Hence the rational points would be $[0 : 1 : u]$, with $u^3 = 1$.

Let X be a cubic curve in $\mathbf{P}^2$. A line intersects X at three points (counted with multiplicity). We define $P * Q$ for any points $P, Q \in X$. If $P \neq Q$, then $P * Q$ is the third point of intersection of X and the line PQ . If $P = Q$, take the tangent T_Q of X at Q then,

$$T_Q \cap X = 2Q + (Q * Q).$$



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