NAKAJIMA QUIVER VARIETIES

RYAN KANNANAIKAL

ABSTRACT. In these notes we will explore some properties of Nakajima quiver varieties. We will also study the action of the framing torus A, on our quiver variety X. In particular how to compute the torus characters of TX at a fixed point $p \in X^A$.

CONTENTS

1. BACKGROUND

If we let Q denote our quiver, then $Q^\sharp$ denotes the framed double of our quiver. A representation of this quiver, with dimension vectors $\mathbf{v}$ and $\mathbf{w}$, is a collection of maps

$$(x, y, I, J) \in \bigoplus_{\alpha \in Q_1} \left(\operatorname{Hom}\left(V_{s(\alpha)}, V_{t(\alpha)}\right) \times \operatorname{Hom}\left(V_{s(\alpha)}, V_{t(\alpha)}\right)^*\right) \times \bigoplus_{i \in Q_0} \left(\operatorname{Hom}\left(W_i, V_i\right) \times \operatorname{Hom}\left(V_i, W_i\right)\right).$$

Since representations of a double quiver correspond to the cotangent bundle of the undoubled quiver, intuitively the maps x, J correspond to the base space, and the maps y, I correspond to the covectors.

We let

$$\mathbf{G}_v = \prod_{i \in Q_0} GL\left(\mathbf{v}_i\right),$$

and similarly we let

$$\mathbf{G}_w = \prod_{i \in Q_0} GL\left(\mathbf{w}_i\right).$$

If we let

$$\mathfrak{g}_v = \prod_{i \in Q_0} \operatorname{Hom}\left(V_i, V_i\right),$$

then our representation space $\mathcal{R}\left(Q^\sharp, v, w\right)$ comes equipped with a symplectic structure, and a moment map $\mu \colon \mathcal{R}\left(Q^\sharp, v, w\right) \to \mathfrak{g}_v$. If we let $Z_v = [\mathfrak{g}_v, \mathfrak{g}_v]^\perp \subset \mathfrak{g}_v^*$ be the fixed points of the coadjoint action. And if we let $\mathcal{Z} = \mu^{-1}(Z_v)$, we define

$$\overline{M}_\theta = \mathcal{Z} /\!\!/_\theta \mathbf{G}_v = \operatorname{Proj} \bigoplus_{n \geq 0} \mathbf{C}[\mathcal{Z}]_{n\theta}.$$

The map μ descends to a map $\overline{\mu}\colon \overline{M}_\theta \to Z_v$.

Definition 1.1. *A Nakajima quiver variety is a fiber of this map, $\mathcal{M}_{\theta,p}(v,w) = \overline{\mu}^{-1}(p)$ for $p \in Z_v$.*

For most of our use we will be working with just the fiber over 0, in this case we let

$$\mathcal{M}_\theta(v,w) = \mu^{-1}(0) /\!\!/_\theta \mathbf{G}_v.$$

Similarly we define the space $\mathcal{M}_0(v,w) = \mu^{-1}(0) /\!\!/ \mathbf{G}_v = \operatorname{Spec} \mathbf{C}\left[\mu^{-1}(0)\right]^{\mathbf{G}_v}$. This construction induces a projection map

$$\pi\colon \mathcal{M}_\theta(v,w) \to \mathcal{M}_0(v,w).$$

2. Stability

Our goal in this section is to relate the notion of (semi)stability in the GIT sense, to an appropriate represenation theoretic notion of stability.

Lemma 2.1. *Let $m = (x,y,I,J) \in \mathcal{R}\left(Q^\sharp, v, w\right)$ and let $\lambda\colon \mathbf{C}^\times \to \boldsymbol{G}_v$ be a one parameter subgroup such that the limit $m_0 = \lim_{t\to 0} \lambda(t).m$ exists. Let*

$$V = \bigoplus_{n \in \mathbf{Z}} V^n$$

be the eigenspace decomposition defined by λ and consider the filtration

$$\cdots V^{\geqslant -1} \supset V^{\geqslant 0} \supset V^{\geqslant 1} \cdots$$

Then

- *Each subspace $V^{\geqslant n}$ is $(x,y)-$invariant*
- $\operatorname{Im} I \subset V^{\geqslant 0}$
- $V^{\geqslant 1} \subset \ker J$.

Definition 2.2. *Let $\theta \in \mathbf{Z}^{Q_0}$, then a representation $(x,y,I,J) \in \mathcal{R}\left(Q^\sharp, v, w\right)$ is called $\theta-$semistable if for any subrepresentation $V' \subset V$, we have*

- $V' \subset \ker J \implies \theta \cdot \dim V' \leq 0$ *resp.*
- $V' \supset \operatorname{Im} I \implies \theta \cdot \dim V' \leq \theta \cdot \dim V$.

Theorem 2.3. *Fix $\theta \in \mathbf{Z}^{Q_0}$, and let χ_θ the corresponding character for $\boldsymbol{G}_v$. Then $m \in \mathcal{R}\left(Q^\sharp, v, w\right)$ is $\chi_\theta-$semistable if and only if the corresponding representation V is $\theta-$semistable.*

Proposition 2.4. *For any $Q, \boldsymbol{v}, \boldsymbol{w}$ there exists a finite set $\{\alpha_i\} \subset \boldsymbol{N}^{Q_0}$ such that $\mathcal{M}_\theta(v,w)$ contains a strictly semistable point only if*

$$\alpha_i \cdot \theta = 0,$$

for some i.

Definition 2.5. *A generic choice of θ, is one that is in the complement of these hyperplanes.*

Theorem 2.6. *Let θ be generic, then every $\theta-$semistable element $m \in \mu^{-1}(0)$ is $\theta-$stable.*

Corollary 2.7. *If θ is generic, and if $\mathcal{M}_\theta(v,w)$ is nonempty, then it is nonsingular.*

Theorem 2.8. *If $\theta > 0$ is generic, then $(x,y,I,J) \in \mathcal{R}\left(Q^\sharp, v, w\right)$ is semistable if and only if for any subrepresentation $V' \subset V$ with $V' \subset \ker J$, then $V = 0$.*

3. Symplectic Resolution

In an appropriate setting, we may interpret a Nakajima quiver variety as a symmplectic resolution of singularities.

Definition 3.1. *A point $m \in \mathcal{R}\left(Q^\sharp, v, w\right)$ is called regular if its orbit is closed and stabilizer is trivial.*

We consider $\mathcal{M}_0^{reg}(v,w) \subset \mathcal{M}_0(v,w)$, the set of regular orbits.

Theorem 3.2. *Let θ be generic, and assume that $\mathcal{M}_0^{reg}(v,w)$ is nonempty. Then*

$$\pi \colon \mathcal{M}_\theta(v,w) \to \mathcal{M}_0(v,w)$$

is a symplectic resolution of singularities.

4. Lagrangians

Since we are in the setting of symplectic geometry, it is important to identify the Lagrangians in the picture. In this section we will identify them as subsets of the so called exceptional fiber. Recall that

$$\mathcal{R}\left(Q^\sharp, v, w\right) = T^*\mathcal{R}\left(\overline{Q}, v, w\right),$$

and so comes with a natural symplectic structure. For some point $(x,y,I,J) \in \mathcal{R}\left(Q^\sharp, v, w\right)$ define a $\mathbf{C}^\times$ action by $\varphi_t(x,y,I,J) = (x, ty, tI, J)$.

Lemma 4.1. *This $\mathbf{C}^\times$action commutes with the $\boldsymbol{G}_v$ action, and*

- $\varphi_t^*(\omega) = t\omega$
- $\mu(\varphi_t(m)) = t\mu\left(m\right)$.

For any θ, the $\mathbf{C}^\times$ action descends to an algebraic action of $\mathbf{C}^\times$ on $\mathcal{M}_\theta(v,w)$. The map $\pi\colon \mathcal{M}_\theta \to \mathcal{M}_0$ is equivariant with respect to this action.

Theorem 4.2.

- *For an $[m] \in \mathcal{M}_0(v,w)$, $\lim_{t\to 0}[m] = 0$.*
- *The only fixed points of $\mathbf{C}^\times$ acting on $\mathcal{M}_0(v,w)$ is $[m] = 0$.*
- *For $[m] \in \mathcal{M}_0(v,w)$ the limit $\lim_{t\to\infty}[m] = 0$ exists only if $[m] = 0$.*

Pick θ generic so that $\mathcal{M}_\theta(v,w)$ is smooth, and let $\mathcal{F} = \mathcal{M}_\theta(v,w)^{\mathbf{C}^\times}$ be the fixed points of this action. By the theorem $\mathcal{F} \subset \pi^{-1}(0)$. Since $\mathcal{M}_\theta(v,w)$ is smooth, and $\mathbf{C}^\times$ is reductive, $\mathcal{F}$ is also smooth. Let $\mathcal{F} = \coprod \mathcal{F}_s$ be the connected components.

Definition 4.3. *The exceptional fiber is $\mathcal{L}\left(v,w\right) = \pi^{-1}(0)$.*

Theorem 4.4.

- *$\mathcal{L}\left(v,w\right) = \bigcup \mathcal{L}_s$, where $\mathcal{L}_s = \left\{m \in \mathcal{M}_\theta(v,w) : \lim_{t\to\infty} t.m \text{ exists and is in } \mathcal{F}_s\right\}$.*
- *Each $\mathcal{L}_s$ is a smooth connected locally closed Lagrangian subvariety in $\mathcal{M}_\theta(v,w)$.*

For $\theta > 0$, and $(x,y,I,J) \in \mu^{-1}(0)$ semistable, then

$$m \in \mathcal{L}\left(v,w\right) \iff I = 0, z = (x,y) \text{ is nilpotent.}$$

5. Tautological Bundles

We fix our dimension vector $\mathbf{v}, \mathbf{w}$ and a generic stability parameter θ. Then for each vertex in our quiver, we get a pair of tautological bundles $\mathcal{V}_i$ and $\mathcal{W}_i$ of ranks $\mathbf{v_i}$ and $\mathbf{w_i}$ respectively on our quiver variety $X = \mathcal{M}_\theta(v, w)$. The bundles $\mathcal{W}_i$ are simply the trivial rank $\mathbf{w_i}$ bundle. Namely $\mathcal{W}_i = X \times W_i$. Now to construct $\mathcal{V}_i$, we note that $\mathbf{G}_v$ acts on $\mu^{-1}(0) \times V_i$ by the diagonal action. Now since $X = \mu^{-1}(0) /\!\!/ \mathbf{G}_v$ is a $\mathbf{G}_v-$quotient, it follows that the induced bundle

$$X \times_{\mathbf{G}_v} V_i$$

is a rank $\mathbf{v_i}$ vector bundle over X.

These tautological bundles are important because all vector bundles on X can be constructed from the tautological bundles using Schur functions. Hence these tautological bundles serve as the building blocks.

6. Torus Fixed Points

There are a number of group actions floating around, so we begin by keeping track of them. To start, we have the action of $\mathbf{G}_v$ on $\mathcal{R}\left(Q^\sharp, v, w\right)$ by $g \cdot (x, y, I, J) = \left(gxg^{-1}, gyg^{-1}, gI, Jg^{-1}\right)$. Similarly we have the action of $\mathbf{G}_w$ on $\mathcal{R}\left(Q^\sharp, v, w\right)$ by $h \cdot (x, y, I, J) = \left(x, y, Ih^{-1}, hJ\right)$. Finally we have the action of $\mathbf{C}_\hbar^\times$ on $\mathcal{R}\left(Q^\sharp, v, w\right)$ by $\hbar \cdot (x, y, I, J) = (x, \hbar y, \hbar I, J)$. Note this last action scales the covector part of $\mathcal{R}\left(Q^\sharp, v, w\right)$. When we pass to the quiver variety X, we are quotienting out the $\mathbf{G}_v$ action, hence can not use it anymore. But the action of $\mathbf{G}_w$ and $\mathbf{C}_\hbar^\times$ are $\mathbf{G}_v-$equivariant hence they descend to the quotient X.

Definition 6.1. *A framing torus A_w is a torus contained in $\boldsymbol{G}_w$ acting on X.*

Not only does our framing torus act on the quiver variety X, it acts on the tautological bundles $\mathcal{V}_i$ and $\mathcal{W}_i$ as follows. Given an element $a \in A$, and $(m, v) \in \mathcal{V}_i$, the action is by $a \cdot (m, v) = (a \cdot m, v)$. Similarly for a point $(m, w) \in \mathcal{W}_i$, the action is $a \cdot (m, w) = (a \cdot m, a \cdot w)$. Note that the torus $A \subset \mathbf{G}_w$ hence it acts on W_i. If we can find a fixed point $p \in X^A$, then we can restrict the action on our tautological bundles to the fibers $\mathcal{V}_i\big|_p$ and $\mathcal{W}_i\big|_p$. Since p is a fixed point, this action preserves the fibers, in other words we get representations of our torus A_w.

A representation for a rank n torus A can be identified with its torus character as follows. If we have a representation V of A, then the vector space V has the following decomposition over characters of A.

$$V = \bigoplus_\chi V_\chi.$$

The characters of A look like maps $\chi\colon (a_1, a_2, \ldots, a_n) \to \prod_i a_i^{\lambda_i}$. Hence we have the correspondence

$$V \leftrightarrow \sum_\chi \dim\left(V_\chi\right) a_I^{\lambda_I} \in \mathbf{C}\left[a_I, a_I^{-1}\right]^+,$$

where $\mathbf{C}\left[a_I, a_I^{-1}\right]^+$ denotes the Laurent polynomials in the a_i with positive coefficients.

Example 6.1. *The character $a_1 + a_2 + a_3$ gets sent to the 3 dimensional representation $\mathbf{C}^3$ with the element (a_1, a_2, a_3) being sent to the matrix*

$$\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}.$$

Similarly $a_1^2 + 2a_3^{-1}$ is the representation coming from the matrix

$$\begin{bmatrix} a_1^2 & & \\ & a_3^{-1} & \\ & & a_3^{-1} \end{bmatrix}.$$

Now if we have some fixed point $p \in X^A$, and some equality $V_1 = V_2$ in $K_A(X)$, then $char_p(V_1) = char_p(V_2)$. So if we can find the characters of our tautological bundles, we can put these together to find the characters at any vector bundle as long as we know how to construct it out of the tautological bundles.

7. Tangent Bundle of Quiver Varieties

As mentioned before all vector bundles on our quiver variety X can be built up from the tautological bundles. The following is the formula for the tangent bundle TX. Letting $K_A(X)$ be the K-group of $A-$equivariant vector bundles on X. We begin by constructing the following virtual bundle,

$$T^{1/2}X = \sum_{i \to j} \operatorname{Hom}(\mathcal{V}_i, \mathcal{W}_j) + \sum_i \operatorname{Hom}(\mathcal{W}_i, \mathcal{V}_i) - \sum_i \operatorname{Hom}(\mathcal{V}_i, \mathcal{V}_i).$$

We let $\hbar^{-1}$ be the trivial line bundle with $A-$action coming from scaling by $\hbar^{-1}$.

Theorem 7.1. *The following formula holds for the class of $TX \in K_A(X)$,*

$$TX = T^{1/2}X + \hbar^{-1} \otimes \left(T^{1/2}X\right)^* \in K_A(X).$$

8. Tensor Product of Quiver Varieties

If we pick a decomposition of $\mathbf{w} = \mathbf{w_1} + \mathbf{w_2}$, then we can construct an action of $\mathbf{C}_b^\times$ as

$$b\,\mathrm{Id}_{w_1} \oplus \mathrm{Id}_{w_2} \subset A_w \subset \mathbf{G}_w,$$

where A_w is the framing torus. If we let

$$\mathcal{M}(w) = \coprod_v \mathcal{M}(v, w),$$

then

$$\mathcal{M}(w)^{\mathbf{C}_b^\times} = \mathcal{M}(w_1) \times \mathcal{M}(w_2).$$

We may also view this decomposition as follows

$$\mathcal{M}(v, w)^{\mathbf{C}_b^\times} = \coprod_{v_1 + v_2 = v} \mathcal{M}(v_1, w_1) \times \mathcal{M}(v_2, w_2).$$

This decomposition is what we denote as the tensor product of quiver varieties.

9. A_1 Quiver

We consider the quiver A_1, which is the quiver with one vertex. If we fix our dimension vector $\mathbf{v} = k$, and $\mathbf{w} = n$, then the representations of the framed double $\mathcal{R}\left(Q^\sharp, k, n\right)$ consists of the maps $(I, J) \in \operatorname{Hom}\left(\mathbf{C}^n, \mathbf{C}^k\right) \times \operatorname{Hom}\left(\mathbf{C}^k, \mathbf{C}^n\right)$.

$$\begin{tikzcd} \bullet \arrow[d, shift right, "J"'] \\ \bullet \arrow[u, shift right, "I"'] \end{tikzcd}$$

If we fix some generic $\theta > 0$ as our stability parameter, then by **Theorem** 2.8 we find that a representation $m = (I, J) \in \mathcal{R}(k, n)$ is (semi)stable, if and only if $\ker J = 0$. Our quiver variety is defined as the following GIT quotient,

$$\mathcal{M}_\theta(v, w) = \mu^{-1}(0) /\!\!/ \mathbf{G}_v.$$

In particular this means, we only want to consider the stable points in the fiber of the moment map. So we want $\mu(m) = -IJ = 0$. Hence the stable representations correspond to the maps (I, J) such that $IJ = 0$, and J is injective. Then the map J is the choice of a k-dimensional subspace $V \subset \mathbf{C}^n$. The fact that $IJ = 0$ means that $I|_V = 0$, and so the map I induces a map $y \colon \mathbf{C}^n/V \to V$. We may canonically identify the collection of these maps as $T_V^* \operatorname{Gr}(k, n)$, and so the corresponding quiver variety, which we denote by $X = T^* \operatorname{Gr}(k, n)$. Similarly the space

$$\mathcal{M}_0(n, k) = \left\{ y \in \operatorname{Hom}\left(\mathbf{C}^2, \mathbf{C}^2\right) : y^2 = 0 \text{ and } \operatorname{rk} y \leq n \right\}.$$

Then the projection map $\pi(y, V) = y$. So we find that the fiber of this projection is $\pi^{-1}(y) = \{(V, y) : \operatorname{Im} y \subset V \subset \ker y\}$.

Let us specialize to the situation when $k = 1$ and $n = 2$, then $\operatorname{Gr}(1, 2) \cong \mathbf{P}^1$, and so $\mathcal{M}_\theta(1, 2) = T^*\mathbf{P}^1$. Furthermore,

$$\mathcal{M}_0(1, 2) = \left\{ x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : x^2 = 0, \operatorname{rk} x \leq 1 \right\}.$$

This is equivalent to $trx = \det x = 0$ which corresponds to matrices

$$\left\{ \begin{bmatrix} a & -b \\ c & -a \end{bmatrix} : -a^2 + bc = 0 \right\}.$$

If we let $Q = \operatorname{Spec} \mathbf{C}[a, b, c]/(bc - a^2)$, this defines a quadric cone in $\mathbf{C}^3$, with a singularity. So the map $\pi \colon T^*\mathbf{P}^1 \to Q$ is the resolution of singularity. This is known as the Springer Resolution.

Let us now calculate the fixed points of the rank n framing torus $A \subset \mathbf{G}_w = \mathbf{GL}_n(\mathbf{C})$. Note that this torus will scale the standard basis vectors e_i of $\mathbf{C}^n$ by a_i. So it is clear that the $k-$planes fixed by this action will be those spanned by k of the standard basis vectors. There are $\binom{n}{k}$ such planes. The fixed points will satisfy the equation

$$(a_1, \ldots, a_n) \cdot (I, J) = g \cdot (I, J)$$

for all $(a_1, \ldots, a_n)$, with $g \in \mathbf{G}_v$. In other words we have the pair of equations

$$IA^{-1} = gI$$
$$AJ = Jg^{-1}.$$

Since J has full rank, it can be put into column echelon form. For the above equations to hold for all choices of A, it follows that $I = 0$. Hence there are $\binom{n}{k}$ fixed points in $T^* \operatorname{Gr}(k, n)$, corresponding to each of the standard $k-$planes in $\operatorname{Gr}(k, n)$ with the cotangent fiber being 0.

Now we will compute the characters of $\mathcal{V}_i$ and $\mathcal{W}_i$ at the corresponding fixed points. Let $S \subset \{1, 2, \ldots, n\}$ be a subset with $\#S = k$. Such subsets will index our fixedpoints. We fix some S corresponding to our fixed point. Since there is only one vertex, our only tautaulogical bundles are $\mathcal{V}$ and $\mathcal{W}$. We begin by calculating the character of $\mathcal{V}$ at the fixed point S. The element

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

acts on the point $[(I, J, v)]$ as $\left[\left(IA^{-1}, AJ, v\right)\right]$. In our case at the fixed point S, $I = 0$. So this simplifies to the point $[(0, AJ, v)]$. Now to calculate the scaling on v, we move our base point back to the representative $(0, J)$ using the group element $g|_S$ which is the diagonal matrix with entries a_i for $i \in S$. Then $g|_S \cdot [(0, AJ, v)] = [(0, J, g|_S\, v)]$. Hence the character of $\mathcal{V}$ at the fixed point S is $\sum_{i \in S} a_i$.

Similarly for $\mathcal{W}$, the element A acts by $A \cdot [(0, I, w)] = [(0, AJ, Aw)]$. Then we bring the representative to the basepoint back with $g|_S$ to find that $g|_S\, [(0, AJ, Aw)] = [(0, J, Aw)]$. So the character of $\mathcal{W}$ at the fixed point S is $\sum_{i=1}^{n} a_i$.

Now let us again specialize to the case when $k = 1$ and $n = 2$ so $X = T^*\mathbf{P}^1$. We will compute the character of A on TX at the fixed point p corresponding to the subset 1. Note that since the J matrix gives our coordinates on the base, this corresonds to the point $([1 : 0], 0) \in \mathbf{P}^1 \times \mathbf{C}$. The other fixed point corresponds to the point $([0 : 1], 0) \in \mathbf{P}^1 \times \mathbf{C}$. Now by **Theorem** 7.1, we have the formula

$$T^{1/2}X = \mathcal{W}^* \otimes \mathcal{V} - \mathcal{V}^* \otimes \mathcal{V}.$$

Furthermore, $TX = T^{1/2}X + \hbar^{-1} \otimes \left(T^{1/2}X\right)^*$. Since the character of $\mathcal{V}$ at p is a_1 and the character of $\mathcal{W}$ at p is $a_1 + a_2$. We have that the character of TX at the point p is given by

$$\begin{aligned} char\left(TX\big|_p\right) &= \left(\frac{1}{a_1} + \frac{1}{a_2}\right) a_1 - \left(\frac{1}{a_1}\right) a_1 + \hbar^{-1}\left((a_1 + a_2)\frac{1}{a_1} - a_1\left(\frac{1}{a_1}\right)\right) \\ &= \frac{a_1}{a_2} + \hbar^{-1}\frac{a_2}{a_1}. \end{aligned}$$

Notice that we have a scaling that has a coefficient of $\hbar^{-1}$. This corresponds to the scaling in the cotangent direction, where as the scaling without the coefficient corresponds to the scaling in the base space direction.

Northeastern University
Email address: `kannanaikal.r@northeastern.edu`