CLASSICAL GROUPS AND GRASSMANNIANS

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ABSTRACT. This is the final paper for Topology 3. We will show that all vector bundles are classified by maps into the infinite Grassmannian. Then characterize the collection of all characteristic classes by the cohomology of Grassmannians. We will conclude by calculating the integral cohomology rings of complex Stiefel manifolds and Grassmannians using the Serre spectral sequence.

CONTENTS

1. INTRODUCTION

We will let $\mathbb{F}$ denote either the real numbers or complex numbers. Note that any finite dimensional vector space over $\mathbb{R}$ and $\mathbb{C}$ are equipped with standard inner products. In this case $\langle x, y\rangle = x^*y$, where x^* is the conjugate transpose of x. In the context of this paper the classical groups will be the general linear groups over $\mathbb{F}$, the subgroups preserving the inner product structure, and the $\det = 1$ subgroups of both.

Example 1.1. Some examples of classical groups include:

(1) $\mathrm{GL}_n(\mathbb{F})$, the general linear group over $\mathbb{F}$.
(2) $U(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^*A = I\}$, the set of unitary matricies over $\mathbb{F}$ which perserve the Hermitian inner product on $\mathbb{C}^n$.
(3) $O(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^TA = I\}$, the set of orthogonal matricies over $\mathbb{R}$ which preserve the usual inner product.
(4) $\mathrm{SL}_n(\mathbb{F})$, the special linear group over $\mathbb{F}$.
(5) $SU(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^*A = I, \det A = 1\}$, the unitary matrices with determinant 1.

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These are all Lie groups, when we consider them with the appropriate topology, and note in particular that $U(n)$ and $SU(n)$ are both compact.

Theorem 1.2 (Theorem 1.1 [BCM12]). *A compact connected Lie group G contains a maximal torus T, any two of which are conjugate.*

2. G-Bundles

Although we will be focusing on vector bundles, it will help to consider the more general setting of fiber bundles. Suppose we have some fiber bundle $\pi\colon E \to B$ with fiber F. Furthermore, suppose that G is some topological group that acts faithfully on F, in other words elements of G may be viewed as elements of $\mathrm{Aut}(F)$. Note that a fiber bundle comes equipped with a local trivialization given by some charts $\{U_i, \varphi_i\}_i$. We may consider the map

$$\varphi_j^{-1}\varphi_i\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

which would send some point (x, α) to (x, β), where α and β are elements of F. We are interested in the case when this mapping is actually an automorphism of the fiber.

Definition 2.1. A $G-$atlas for a fiber bundle is a local trivialization, $\{(U_i, \varphi_i)\}_i$ such that $\varphi_j^{-1}\varphi_i(x, \alpha) = (x, t_{ij}(x)\alpha)$, where $t_{ij}\colon U_i \cap U_j \to G$ is a continuous map, called the transition function. Two $G-$atlases are equivalent if their union is also a $G-$atlas. A $G-$bundle is a fiber bundle with an equivalence class of $G-$atlases. We call the group G the structure group of the bundle.

Notice that the transition functions satisfy the following conditions:

(1) $t_{ii}(x) = 1$
(2) $t_{ij}(x) = t_{ji}(x)^{-1}$
(3) $t_{ik}(x) = t_{ij}(x)t_{jk}(x)$.

Definition 2.2. A morphism of $G-$bundles $(\tilde{f}, f)\colon \xi \to \xi'$, is a pair of maps $f\colon B \to B'$, and $\tilde{f}\colon E \to E'$, that fit into the following commutative diagram,

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

and $\tilde{f}\Big|_{x \times F}\colon F \to F$ is $G-$equivariant over each point $x \in X$.

Theorem 2.3 (Theorem 1.5 [BCM12]). *If G acts faithfully on F, then there exists one and, up to equivalence, only one $G-$bundle with fiber F base space B, and given system of transition functions t_{ij}. Furthermore, two $G-$bundles ξ and ξ' are equivalent if and only if there exists a map $\psi_i\colon U_i \to G$ such that*

$$t'_{ij}(x) = \psi_j^{-1}(x)t_{ij}(x)\psi_i(x).$$

In other words, this theorem tells us that the data of the transition functions is enough to uniquely determine the $G-$bundle.

Definition 2.4. A principal $G-$bundle is a $G-$bundle with fiber G and $G-$action defined by $g \cdot g' = gg'$. See [McC01] for more details.

Definition 2.5. For any $G-$bundle ξ, we can construct the associated principle bundle, $\text{Prin}(\xi)$ by picking a cover subordinate to that of ξ, and then defining the $t'_{ij} \colon U_i \cap U_j \to G$ to be the same as the original transition functions.

The fact that $\varphi_j^{-1}\varphi_i(x, \gamma) = (x, t_{ij}(x)\gamma)$ makes sense since $t_{ij}(x)$ and γ are both elements of G so can be multiplied.

3. Homogeneous Spaces

Recall that if H is a closed subgroup of a topological group G, then we can view G/H as the space of left cosets of H equipped with the quotient topology. This coset space is called a homogeneous space. Suppose that G acts transitively on some space X, and Gx is the isotropy group of some $x \in X$. Then Gx is a closed subgroup of G, and $G/Gx \cong X$ if G is compact and Hausdorff.

Definition 3.1. A k-frame on $\mathbb{F}^n$ is an ordered k-tuple of orthonormal vectors in $\mathbb{F}^n$. A k-plane on $\mathbb{F}^n$ is a subspace of dimension k. Let $V_k(\mathbb{F}^n)$ be the set of k-frames on $\mathbb{F}^n$, along with the subspace topology inherited from $\mathbb{F}^n$. Likewise, let $G_k(\mathbb{F}^n)$ be the set of k-planes on $\mathbb{F}^n$, adopting the quotient topology under the map $\text{span} \colon V_k(\mathbb{F}^n) \to G_k(\mathbb{F}^n)$, sending the k vectors to their span. We call $V_k(\mathbb{F}^n)$ and $G_k(\mathbb{F}^n)$ the Stiefel and Grassmann manifolds on $\mathbb{F}^n$ respectively.

Notice that $U(n)$ acts transitively on $V_k(\mathbb{C}^n)$ and $G_k(\mathbb{C}^n)$. Suppose that $x \in V_k(\mathbb{C}^n)$, then $U(n)x \cong U(n-k)$ since the isotropy group of x consists of all unitary matrices that fix the k vectors in x, but permute the remaining $n-k$.

Likewise if $X = \text{span}(x)$, then the isotropy group of X is $U(k) \times U(n-k)$ since any unitary matrix fixing a $k-$plane must fix its orthogonal complement. It follows that $V_k(\mathbb{C}^n) = U(n)/U(n-k)$ and $G_k(\mathbb{C}^n) = U(n)/(U(n-k) \times U(k))$, that is these two manifolds are homogeneous spaces. We can also view $V_k(\mathbb{R}^n)$ and $G_k(\mathbb{R}^n)$ as homogeneous spaces, by replacing the unitary groups by the orthogonal groups. From now on we will only consider the complex case.

Notice that $U(k)$ acts freely on $V_k(\mathbb{C}^n)$, and so the above calculation shows that

$$V_k(\mathbb{C}^n)/U(k) = G_k(\mathbb{C}^n).$$

4. Classifying Spaces

Definition 4.1. A universal bundle for a topological group G is a principal $G-$bundle $p \colon EG \to BG$ such that EG is contractible. Any such base space BG is called a classifying space of G.

Theorem 4.2 (The Classification Theorem [Ste51])**.** *If G acts freely on a space F, then equivalence classes of $G-$bundles over X with fiber F are in natural one to one correspondence with homotopy classes of maps $X \to BG$, denoted $[X, BG]$.*

Under this correspondence, we associate to a map $f\colon X \to BG$ the $G-$bundle with fiber F associated to the induced principal bundle f^*p.

Definition 4.3. A characteristic class c associates to each $G-$bundle ξ over X, a cohomology class $c(\xi) \in H^*(X)$. This association is natural, with respect to maps of $G-$bundles. That is, if $(\tilde{f}, f)\colon \xi \to \xi'$ is a morphism of $G-$bundles, then $f^*c(\xi') = c(\xi)$.

Lemma 4.4. *Characteristic class for $G-$bundles are in one to one correspondence with elements of $H^*(BG)$.*

Proof. We may assume that our bundle is principal, by passing to the associated principal bundle. Since G acts freely on G, we may use Theorem 4.2 to derive a correspondence between equivalence classes of $G-$bundles over X with $[X, BG]$. Then a characteristic class c is a natural transformation between the functors $[?, BG] \to H^*(?)$. By the Yoneda lemma this is in one to one correspondence with $H^*(BG)$. ☐

5. Classifying space of $\mathrm{GL}_n(\mathbb{C})$

If we let $p\colon E \to B$ be a rank n vector bundle, then we may consider the associated principal $\mathrm{GL}_n(\mathbb{C})$-bundle. Since this associated bundle admits local sections, we get a local trivialization of our original bundle. Similarly, if we have a principal $\mathrm{GL}_n(\mathbb{C})$-bundle $p\colon E \to B$, we can defined a rank n vector bundle $V = P \times \mathbb{R}^n/G$. The right action of $\mathrm{GL}_n(\mathbb{C})$ is defined as $(x, v) \cdot g = (x \cdot g, g^{-1} \cdot v)$. Hence we may view a complex vector bundle as a $\mathrm{GL}_n(\mathbb{C})-$bundle, in particular characteristic classes are in one to one correspondence with elements of $H^*(B\,\mathrm{GL}_n(\mathbb{C}))$.

Proposition 5.1. *The classifying space $B\,\mathrm{GL}_n(\mathbb{C})$ is homotopy equivalent to $BU(n)$.*

Proof. Pick any matrix $A \in \mathrm{GL}_n(\mathbb{C})$, and consider its QR decomposition, $A = QR$. Since Q is unitary, we make pick it so that $\det Q = 1$. Furthermore, since R is upper triangular, it can clearly be deformation retracted to the identity through matrices with nonzero determinant. Hence the inclusion of $U(n)$ into $GL_n(\mathbb{C})$ is a deformation retract, and so $BU(n) \simeq B\,\mathrm{GL}_n(\mathbb{C})$. ☐

So we wish to find $BU(n)$. Considering the fibration

$$V_{k-1}(\mathbb{C}^{n-1}) \longrightarrow V_k(\mathbb{C}^n) \longrightarrow S^{2n-1}$$

and then looking at the long exact sequence in homotopy gives us the fact that

$$\pi_i(V_k(\mathbb{C}^n)) = \pi_i(V_{k-1}(\mathbb{C}^{n-1})) = \cdots = \pi_i(\mathbb{C}^{n-k+1})) = \pi_i(S^{2n-2k+1}).$$

Hence we conclude that $V_k(\mathbb{C}^n)$ is $(2n-2k+1)$-connected. Since the Stiefel manifold is a CW complex, it follows that the infinite Stiefel manifold $V_k = \varinjlim_n V_k(\mathbb{C}^n)$ is nullhomotpic by

Whitehead's theorem. Note $U(n)$ acts freely on V_k, and $\varinjlim_n V_k(\mathbb{C}^n)/U(n) = \mathrm{Gr}_k$, where Gr_k is the infinite Grassmannian. This means that $\pi\colon V_k \to \mathrm{Gr}_k$ is a principle $U(n)$-bundle, and since V_k is contractible, it follows that $\mathrm{Gr}_k = BU(n)$.

6. Cohomology of Stiefel Manifolds

We have already shown that the Stiefel manifold $V_k(\mathbb{C}^n) = U(n)/U(n-k)$. Similarly, if we consider the action of $SU(n)$ on $V_k(\mathbb{C}^n)$ any point has the isotropy group $SU(n-k)$. This is because we can extend any $k-$frame to an $n-$frame on $\mathbb{C}^n$ be adding $n-k$ new vectors, and then any element of $SU(n-k)$ permuting these last elements will fix x. It follows that $V_k(\mathbb{C}^n) = SU(n)/SU(n-k)$. Hence we get the following fibration,

$$SU(n-k) \longrightarrow SU(n) \longrightarrow V_k(\mathbb{C}^n).$$

Now if we consider the inclusion of $SU(n-k-1)$ into $SU(n-k)$, we get the new fibration,

$$SU(n-k)/SU(n-k-1) \longrightarrow SU(n)/SU(n-k-1) \longrightarrow SU(n)/SU(n-k-1).$$

If we identify $SU(n-k)/SU(n-k-1)$ with $S^{2(n-k)-1}$ the remaining two terms with the appropriate Stiefel manifolds, we get the following fibration,

$$S^{2(n-k)-1} \longrightarrow V_{k+1}(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n).$$

Proposition 6.1 (Proposition 5.11 [McC01])**.**

$$H^*(V_k(\mathbb{C}^n)) \cong \Lambda[x_{2(n-k)+1}, x_{2(n-k)+3}, \cdots, x_{2n-1}].$$

Proof. We will induct on k. Suppose $k = 1$, then $H^*(V_1(\mathbb{C}^n)) = H^*(S^{2n-1}) = \Lambda[x_{2n-1}]$, hence the proposition holds. Now suppose it holds for k, we show it holds for $k+1$. Then the Serre spectral sequence is given by,

$$E_2^{p,q} = H^p(V_k(\mathbb{C}^n); H^q(S^{2(n-k)-1})) \implies H^{p+q}(V_{k+1}(\mathbb{C}^n)).$$

By the Universal coefficient theorem, and the fact that the generators form a simple system, the E_2 page is the tensor product of the 0-row and 0-column. In other words $E_2^{p,q} \cong \Lambda[x_{2(n-k)+1}, x_{2(n-k)+3}, \cdots, x_{2n-1}] \otimes \Lambda[y_{2(n-k)-1}]$ by the inductive hypothesis. Note that the most a differential could lower the q-degree is $2(n-k)-1$, before it has negative degree. In other words the last possible page where anything could change is $E_{2(n-k)}$. However on this page the $d_{2(n-k)}$ differential only goes from i to $i+2n-2k$, which is not enough to reach the next terms, and so even on this page all differential are 0. This means that $E_2 = E_\infty$, so

$$H^*(V_{k+1}(\mathbb{C}^n)) = \bigoplus_{p+q=n} E_2^{p,q} = \Lambda[y_{2(n-k)-1}, x_{2(n-k)+1}, x_{2(n-k)+3}, \cdots, x_{2n-1}].$$

$\square$

7. Cohomology of Infinite Grassmannian

We will conclude by computing the cohomology ring of the infinite complex Grassmannian, which again classifies all characteristic classes of complex vector bundles. We start by computing the cohomology ring of the Unitary group.

Lemma 7.1. *The cohomology of $U(n)$ is*

$$H^*(U(n)) \cong \Lambda[x_1, x_3, x_5, \cdots, x_{2n-1}]$$

where the degree is the index of the generator.

Proof. We will induct on n. When $n = 1$, then $H^*(U(1)) \cong H^*(S^1) \cong \Lambda[x_1]$, and so the base case holds. Supposing it holds for $n-1$, we consider the fiber bundle

$$U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}$$

and then use the Serre spectral sequence. Then the E_2 page has entries in the columns with index 0 and $2n-1$. If we let α be the generator of $H^{2n-1}(S^{2n-1})$, then $E_2^{0,q} = H^q(U(n-1); \mathbb{Z}) \otimes \mathbb{Z}\{1\}$, and $E_2^{2n-1,q} = H^q(U(n-1); \mathbb{Z}) \otimes \mathbb{Z}\{\alpha\}$. By assumption the highest degree generator in $H^*(U(n-1))$ is x_{2n-3} of degree $2n-3$. The only nonzero differential would be d_{2n-1}, but then it will reduce the degree of q by $2n-2$, and so even the highest degree generator will be in negative degree. It follows that $E_2 = E_\infty$, and so by a similar argument to the proof of Proposition 6.1, the conclusion follows. □

Theorem 7.2. *The cohomology of the infinite Grassmannian* Gr_k *is,*

$$H^*(\mathrm{Gr}_k) \cong \mathbb{Z}[c_1, \cdots, c_k]$$

where $|c_i| = 2i$.

Proof. We consider the following fiber bundle

$$U(k) \longrightarrow V_k \longrightarrow \mathrm{Gr}_k$$

and use the Serre spectral sequence. The result will follow from Theorem 3.27 in [McC01]. We must check that the hypotheses are met. Since V_k is contractible, it follows that $E_\infty^{0,0} = K$ for some field K, and $E_\infty^{p,q} = 0$. The above Lemma 7.1 shows the second condition, namely that the cohomology of the fiber is an exterior algebra on odd degree generators. Suppose that the generators x_i were not transgressive,then d_{2r_i} would have a nontrivial kernel, and so an element on E_{2r_i+1} would persist to E_∞, but this is not possible, hence the last condition is met. So $H^*(\mathrm{Gr}_k) \cong K[c_1, \cdots, c_k]$, where $c_i = d_{2r_i}(x_i)$. Finally we use the Universal coefficient theorem to pass from coefficients over the field K to coefficients over $\mathbb{Z}$. □

References

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