# FOURIER FUNCTOR

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## 1. Introduction

This is a very short supplement/summary to section 2 in [Muk81]. Let $A$ be an abelian variety of dimension $g$, and let $\hat{A}$ its dual abelian variety. We let $\mathscr{P}$ be the normalized Poincaré bundle on $A \times \hat{A}$, i.e. $\mathscr{P}\big|_{A\times\hat{0}}$ and $\mathscr{P}\big|_{0\times\hat{A}}$ are trivial. We let $\mathsf{R}\varphi(\mathcal{F}) = \mathsf{R}\pi_{A,*}(\mathscr{P} \overset{L}{\otimes} \pi_{\hat{A}}^{*}\mathcal{F})$, and $\mathsf{R}\hat{\varphi}(\mathcal{F}) = \mathsf{R}\pi_{\hat{A},*}(\mathscr{P} \overset{L}{\otimes} \pi_{A}^{*}\mathcal{F})$.

## 2. Results

**Theorem 2.1.** [Muk81] *There are isomorphisms of functors:*

$$R\varphi \circ R\hat{\varphi} \cong (-1_A)^*[-g]$$

*and*

$$R\hat{\varphi} \circ R\varphi \cong (-1_{\hat{A}})^*[-g].$$

**Definition 2.2.** Let $\mathcal{F}$ a coherent sheaf on $A$. The Weak Index Theorem (W.I.T) hold for $\mathcal{F}$ if $\mathsf{R}^i\hat{\varphi}(\mathcal{F}) = 0$ for all but one $i$. We call this $i$ the index of $\mathcal{F}$, denoted $i(\mathcal{F})$.

**Definition 2.3.** We let $\hat{\mathcal{F}} \overset{\text{def}}{=} \mathsf{R}^{i(\mathcal{F})}\hat{\varphi}(\mathcal{F})$ and call it the Fourier transform of $\mathcal{F}$.

**Definition 2.4.** We say that the Index Theorem (I.T.) holds for $\mathcal{F}$ if $H^i(A, \mathcal{F} \otimes P) = 0$ for all $P \in \operatorname{Pic}^0(A)$ and all but one $i$.

**Proposition 2.5.** *We have that*

- *Index theorem* $\implies$ *Weak index theorem*
- $\hat{\mathcal{F}}$ *is locally free if I.T. holds for* $\mathcal{F}$.

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*Proof.* We will prove this by cohomology and base change, which is Theorem 12.11 in [Har77]. We consider the natural map

$$\Theta^i\colon \mathsf{R}^i\pi_{\hat{A},*}(\mathscr{P}\overset{L}{\otimes}\pi_A^*\mathcal{F})\otimes k(\hat{a})\to H^i\left(A\times\hat{A}\Big|_{A\times\hat{a}},(\mathscr{P}\overset{L}{\otimes}\pi_A^*\mathcal{F})\Big|_{A\times\hat{a}}\right)$$

and wish to show that it is surjective. However, $(\mathscr{P}\overset{L}{\otimes}\pi_A^*\mathcal{F})\Big|_{A\times\hat{a}}\cong P_{\hat{a}}\otimes\mathcal{F}$, so the target of $\Theta^i$ is $H^i(A_{\hat{a}},P_{\hat{a}}\otimes\mathcal{F})=0$. □

If we identify $\mathcal{F}$ with the complex concentrated in degree 0, it follows that if W.I.T. holds for $\mathcal{F}$ then $\mathsf{R}\hat{\varphi}(\mathcal{F})$ is isomorphic to $\hat{\mathcal{F}}[-i(\mathcal{F})]$.

**Corollary 2.6.** *If W.I.T. holds for $\mathcal{F}$, then so does for $\hat{\mathcal{F}}$ and $i(\hat{\mathcal{F}})=g-i(\mathcal{F})$. Moreover $\hat{\hat{\mathcal{F}}}$ is isomorphic to $(-1_A)^*\mathcal{F}$.*

*Proof.* Letting $\hat{\mathcal{F}}=\mathsf{R}^{i(\mathcal{F})}\hat{\varphi}(\mathcal{F})$, we wish to compute $\mathsf{R}^j\varphi(\hat{\mathcal{F}})$. This is then $\mathsf{R}^j\varphi(\mathsf{R}^{i(\mathcal{F})}\hat{\varphi}(\mathcal{F}))$. Since we have a composition of two derived functors we will use the Grothendieck spectral sequence to compute the composition. Recall that

$$E_2^{p,q}=\mathsf{R}^pF(\mathsf{R}^qG(X))\implies\mathsf{R}^{p+q}(F\circ G)(X).$$

Then the $E_2$ page has terms only in row $i(\mathcal{F})$, it follows that $E_2=E_\infty$. Note also that $\mathsf{R}^{j+i(\mathcal{F})}(\varphi\circ\hat{\varphi})=\mathsf{R}^{j+i(\mathcal{F})}(-1_A)^*[-g]=\mathsf{R}^{j+i(\mathcal{F})-g}(-1_A)^*$, and since $(-1_A)^*$ is exact it follows that the higher derived functors vanish, so the only nonzero entry is when $j+i(\mathcal{F})-g=0$, and since $E_2=E_\infty$, it follows that only $\mathsf{R}^{g-i(\mathcal{F})}(\hat{\mathcal{F}})\neq 0$.

Finally consider the following computation,

$$\begin{aligned}\hat{\hat{\mathcal{F}}}&=\mathsf{R}^{g-i(\mathcal{F})}\varphi(\mathsf{R}^{i(\mathcal{F})}\hat{\varphi}\mathcal{F})\\&=\mathsf{R}^g(\varphi\circ\hat{\varphi})\mathcal{F}\\&=\mathsf{R}^g(-1_A)^*[-g]\mathcal{F}\\&=\mathsf{R}(-1_A)^*\mathcal{F}=(-1_A)^*\mathcal{F}.\end{aligned}$$

□

**Corollary 2.7.** *Assume that W.I.T. holds for $\mathcal{F}$ and $\mathcal{G}$. Then $\operatorname{Ext}^i(\mathcal{F},\mathcal{G})\cong\operatorname{Ext}^{i+\mu}(\hat{\mathcal{F}},\hat{\mathcal{G}})$ for every integer $i$, where $\mu=i(\mathcal{F})-i(\mathcal{G})$.*

*Proof.*

$$\begin{aligned}\operatorname{Ext}^i(\mathcal{F},\mathcal{G})&\cong\operatorname{Hom}_{\mathbf{D}(A)}(\mathcal{F},\mathcal{G}[i])\\&\cong\operatorname{Hom}_{\mathbf{D}(\hat{A})}(\mathsf{R}\hat{\varphi}(\mathcal{F}),\mathsf{R}\hat{\varphi}(\mathcal{G})[i])\\&\cong\operatorname{Hom}_{\mathbf{D}(\hat{A})}(\hat{\mathcal{F}}[-i(\mathcal{F})],\hat{\mathcal{G}}[i-i(\mathcal{G})])\\&\cong\operatorname{Hom}_{\mathbf{D}(\hat{A})}(\hat{\mathcal{F}},\hat{\mathcal{G}}[i+i(\mathcal{F})-i(\mathcal{G})])\\&\cong\operatorname{Ext}^{i+\mu}(\hat{\mathcal{F}},\hat{\mathcal{G}}).\end{aligned}$$

The first isomorphism is by the properties of the derived category, the second is by the equivalence of derived categories. $\square$

**Proposition 2.8.** *Assume that W.I.T. holds for a coherent sheaf $\mathcal{F}$ on $A$. Then we have*

$$H^i(A, \mathcal{F} \otimes P_{\hat{x}}) \cong \operatorname{Ext}^{g-i(\mathcal{F})+i}(k(\hat{x}), \hat{\mathcal{F}})$$

*and*

$$\operatorname{Ext}^i(k(x), \mathcal{F}) \cong H^{i-i(\mathcal{F})}(\hat{A}, \hat{\mathcal{F}} \otimes P_{-x}).$$

*Proof.* By Corollary 2.6, it suffices to show the first isomorphism. Since $P_{\hat{x}}$ is locally free, we have that

$$\begin{aligned} H^i(X, \mathcal{F} \otimes P_{\hat{x}}) &\cong \operatorname{Ext}^i(\mathcal{O}_A, \mathcal{F} \otimes P_{\hat{x}}) \\ &\cong \operatorname{Ext}^i(P_{-\hat{x}} \otimes \mathcal{O}_A, \mathcal{F} \otimes P_{\hat{x}} \otimes P_{-\hat{x}}) \\ &\cong \operatorname{Ext}^i(P_{-\hat{x}}, \mathcal{F}). \end{aligned}$$

We get the second line by tensoring both sides by the locally free $P_{-\hat{x}}$. Hence by Corollary 2.7, it is isomorphic to $\operatorname{Ext}^{i+\mu}(\widehat{P}_{-\hat{x}}, \hat{\mathcal{F}})$. Let $k(-\hat{x})$ be the skyscrapper sheaf supported at $\hat{x} \in \hat{A}$, we have computed that $\widehat{k(-\hat{x})} = P_{-\hat{x}}$. It follows that $\widehat{P_{-\hat{x}}} = \widehat{\widehat{k(-\hat{x})}} = (-1_{\hat{A}})^* k(-\hat{x}) = k(\hat{x})$. Hence $\operatorname{Ext}^{i+\mu}(\widehat{P}_{-\hat{x}}, \hat{\mathcal{F}}) \cong \operatorname{Ext}^{i+\mu}(k(\hat{x}), \hat{\mathcal{F}})$, where $\mu = i(P_{-\hat{x}}) - i(\mathcal{F}) = g - i(\mathcal{F})$. $\square$

**Corollary 2.9.** *The Euler-Poincaré characteristic of $\mathcal{F}$ is equal to $(-1)^{i(\mathcal{F})} r(\mathcal{F})$.*

*Proof.*

$$\begin{aligned} \chi(X, \mathcal{F}) &= \sum_i (-1)^i h^i(X, \mathcal{F}) \\ &= \sum_i (-1)^i \dim \operatorname{Ext}^{i+g-i(\mathcal{F})}(k(\hat{x}), \hat{\mathcal{F}}). \end{aligned}$$

Suppose that $\hat{\mathcal{F}}$ is locally free, then by Grothendieck-Verdier duality we have that $\operatorname{Ext}^{i+g-i(\mathcal{F})}(k(\hat{x}), \hat{\mathcal{F}}) \cong \operatorname{Ext}^{i(\mathcal{F})-i}(\hat{\mathcal{F}}^\vee, k(\hat{x}))$, which is isomorphic to $H^{i(\mathcal{F})-i}(\hat{\mathcal{F}}^\vee \otimes k(\hat{x}))$. Since we are tensoring with the structure sheaf supported at a point, it follows that we only have nonzero cohomology when $i(\mathcal{F}) - i = 0$. Then we have that

$$\begin{aligned} \dim H^0(\hat{\mathcal{F}}^\vee \otimes k(\hat{x})) &= \dim H^0(\hat{\mathcal{F}}^\vee \big|_{\hat{x}}) \\ &= r(\hat{\mathcal{F}}^\vee) \\ &= r(\hat{\mathcal{F}}). \end{aligned}$$

Supposing that $\hat{\mathcal{F}}$ is not locally free, we may take a locally free resolution of $\hat{\mathcal{F}}$, and the same computation will show that the above sum returns $(-1)^{i(\mathcal{F})} r(\hat{\mathcal{F}})$. $\square$

## References

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1977.

[Muk81] Shigeru Mukai, *Duality between* $\mathbf{D}(X)$ *and* $\mathbf{D}(\hat{X})$ *with its application to picard sheaves*, Nagoya Math. J. (1981).

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