DERIVED CATEGORIES AND DERIVED FUNCTORS

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Contents

1. Introduction

This is a very short supplement/summary to sections 2.1 and 2.2 in [Huy06].

2. Derived Category

2.1. **Quasi-isomorphism.** Why do we care about quasi-isomorphism (qis.)? Well one reason is because we like injective/projective resolutions.

Example 2.1. Consider the following projective resolution of M,

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

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this resolution gives rise to the following qis.

$$\begin{tikzcd} \cdots \arrow[r] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

The only place with nontrivial cohomology is in degree 0, and in both cases the cohomology is M. Note that $P^0/\operatorname{img}(P^{-1} \to P^0) \cong P^0/\ker(f)$ by exactness, and the isomorphism with M follows since f is surjective.

So if we "turn" quasi-isomorphisms into isomorphisms we can replace M by this projective resolution which is much easier to work with. Dually, we can work with injective resolutions which will probably be more common in the categories we work with.

Example 2.2. We can define two truncation functors, which will be useful to keep in the back pocket. Given some complex $A^\bullet$, let $\tau_{\leq 0}(A^\bullet)$ denote the following truncated complex.

$$\cdots \longrightarrow A^{-1} \longrightarrow \ker d \longrightarrow 0 \longrightarrow \cdots$$

In this case d denotes the differential of the complex $d\colon A^0 \to A^1$. We have the inclusion $\iota\colon \tau_{\leq 0}(A^\bullet) \to A^\bullet$. Also by construction $H^i(\tau_{\leq 0}(A^\bullet)) \cong H^i(A^\bullet)$ for $i \leq 0$, and $H^i(\tau_{\leq 0}(A^\bullet)) = 0$, for $i > 0$. In particular if $H^i(A^\bullet) = 0$ for $i > 0$ then ι is a qis.

Likewise, let $\tau_{\geq 0}(A^\bullet)$ denote the following truncated complex.

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{coker} d \longrightarrow A^1 \longrightarrow \cdots$$

Similar properties hold for this complex as well.

2.2. **Cohomology Functor in K(A).** A quick note, the cohomology functor, $H^n\colon \mathcal{A} \to \mathcal{A}$ descends to the homotopy category $K(\mathcal{A})$. This functor is well defined since two homotopic maps induce the same morphism on cohomology.

Proof. If $f \sim g$, then $f - g = dh + hd$ where d denotes the differential and h the homotopy. We apply this difference to some element $\sigma \in \ker d$. So the $(f-g)\sigma = d(h\sigma)$, since $hd(\sigma) = 0$. In particular $f - g \in \operatorname{img}(d)$ so the difference is 0 in cohomology. □

Corollary 2.3. *If $f\colon A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a qis.*

2.3. **Localization.** Similarly to how localizing a multiplicative set in a commutative ring formally inverts those elements with respect to multiplication, we can formally invert morphisms, if they form a localizing class.

Definition 2.4. A class of morphisms S in some category $\mathcal{A}$ is a localizing class if they satisfy the following conditions.

- S is closed under composition, and contains all identity maps.

- (Ore Condition) The following two diagrams can be completed.

$$\begin{tikzcd} T \arrow[r, dotted] \arrow[d, dotted, "s' \in S"'] & X \arrow[d, "s \in S"] \\ X \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s \in S"'] & Y \arrow[d, dotted, "s' \in S"] \\ Z \arrow[r, dotted] & T \end{tikzcd}$$

- If $f, g\colon X \to Y$ are parallel maps in $\mathcal{A}$ then the following two conditions are equivalent
  (1) $sf = sg$ for some $s \in S$
  (2) $ft = gt$ for some $t \in S$.

The upshot for introducing this definition is to motivate some of what happens in section 2.1 of the text. The goal is more or less to establish the fact that the class of quasi-isomorphisms in $K(\mathcal{A})$ form a localizing class. For example Proposition 2.17 in [Huy06] is checking the Ore condition. So $D(\mathcal{A})$ is precisely the localization of the homotopy category, $K(\mathcal{A})[Q^{-1}]$, where Q is the class of quasi-isomorphisms.

2.4. **Calculus of Roofs.** When we localize a ring, there is a "calculus of fractions" that lets us keep track of how different equivalence classes behave. The very same thing happens, where in the localized category the morphisms are "roof" diagrams..

$$\begin{tikzcd} & C \arrow[dl, "qis"'] \arrow[dr] & \\ A & & B \end{tikzcd}$$

Note that we get composition of roofs thanks to the Ore condition.

$$\begin{tikzcd} & & Z \arrow[dl, dotted, "qis"'] \arrow[dr] & & \\ & X \arrow[dl, "qis"'] \arrow[dr] & & Y \arrow[dl, "qis"'] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

This is Corollary 2.18 in the book. We said that morphisms in the localizing class become invertible. So let's quickly check that a qis $s\colon A \to B$ becomes invertible in $D(\mathcal{A})$.

Proof. We will compose the following two maps.

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}"'] \arrow[dr, "s"] & \\ A & & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\ B & & A \end{tikzcd}$$

Their composition becomes the following roof,

$$\begin{tikzcd}
 & & A \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & \\
 & A \arrow[dl, "\mathrm{id}"'] \arrow[dr, "s"] & & A \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\
A & & B & & A
\end{tikzcd}$$

and this is $\mathrm{id}_A \in \mathrm{Hom}_{D(\mathcal{A})}(A, A)$, the other direction is similar. □

Remark 2.5. The comology functor factors through the Derived category, that is we have the following diagram.

$$\begin{tikzcd}
H^n\colon K(\mathcal{A}) \arrow[rr] \arrow[dr] & & \mathcal{A} \\
 & D(\mathcal{A}) \arrow[ur, dotted] &
\end{tikzcd}$$

This is because a diagram of this form,

$$\begin{tikzcd}
 & C \arrow[dl, "qis"'] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

gives us the following diagram once we apply H^n and note that qis are sent to isomorphisms by the cohomology functor.

$$\begin{tikzcd}
 & H^nC \arrow[dl, "\cong"'] \arrow[dr] & \\
H^nA & & H^nB
\end{tikzcd}$$

We get the map from $H^nA \to H^nB$ by going along the top of the diagram through the inverse of the isormophism.

2.5. **Mapping Cone.**

Definition 2.6. Given a map $f\colon A^\bullet \to B^\bullet$, the mapping cone $C(f) = A[1] \oplus B$ with differential

$$\begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}$$

Remark 2.7. We note that $H^{-1}(C(f)) = \ker f$, and $H^0(C(f)) = \operatorname{coker} f$.

So why do we care about the mapping cone? For starters it lets us define the distinguished triangles in $K(\mathcal{A})$ and $D(\mathcal{A})$, but also it can be used to check if a map is a qis. We have the following s.e.s of complexes.

$$0 \longrightarrow B \longrightarrow C(f) \longrightarrow A[1] \longrightarrow 0$$

So we get the following long exact sequence in cohomology.

$$\cdots \longrightarrow H^iA \longrightarrow H^iB \longrightarrow H^iC(f) \longrightarrow H^{i+1}A \longrightarrow \cdots$$

Corollary 2.8. *A map $f\colon A \to B$ is a qis if and only if $H^*(C(f)) = 0$. That is if $C(f)$ is* acyclic.

Remark 2.9. Also note this means $C(f) \cong 0$ in $D(\mathcal{A})$.

2.6. **Question on Proposition 2.17.** ???Was not able to show that $\varphi\colon C_0^\bullet \to C^\bullet$ is a qis. But my guess is we show somehow that $C(\varphi) \cong C(f) \cong 0$???

Remark 2.10. $D(\mathcal{A})$ is additive, we can add two maps

$$\begin{tikzcd} & X \arrow[dl] \arrow[drr] & Y \arrow[dll] \arrow[dr] & \\ A & & & B \end{tikzcd}$$

by completing them to

$$\begin{tikzcd} & Z \arrow[dl, "qis"'] \arrow[dr, "qis"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd}$$

then adding the two maps into B

$$\begin{tikzcd} & Z \arrow[dl, "qis"'] \arrow[dr, bend left] \arrow[dr, bend right] & \\ A & & B \end{tikzcd}$$

2.7. **Injectives and Projectives.**

Definition 2.11. An object P is projective if it satisfies the following diagram.

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

An object I is injective if it satisfies the dual diagram.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dotted] \\ I & \end{tikzcd}$$

Remark 2.12. The category of quasi-coherent sheaves over a noetherian scheme X has enough injectives. III Corollary 3.6 in [Har77]. The idea is that such a sheaf $\mathscr{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathscr{G}$, and this sheaf $\mathscr{G}$ also happens to be an injective object. The construction basically uses the fact that the category of modules has enough injectives and glues things together nicely over an open cover of X.

2.8. **Hom sets in D(A).**

Theorem 2.13. *Let $\mathcal{A}$ have enough injectives, and $A, B \in \mathcal{A}$. Then*

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]).$$

Why should we care about this? Well it gives us a hint that all the important algebra in the higher Ext groups is captured by morphisms between very natural looking complexes in $D(\mathcal{A})$. In other words, understanding $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ will tell us a lot of good information. But computing in the derived category is not easy. This is the importance of Prop 2.35 - Prop 2.40, it will let us do computations in $K(\mathcal{A})$ to understand Hom sets in $D(\mathcal{A})$. Prop 2.40 concludes that $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is an equivalence of categories.

2.9. **Proposition 2.35.** If $\mathcal{A}$ has enough injectives and $A^\bullet$ is bounded below then it is qis to $I^\bullet \in K^+(\mathcal{A})$.

2.10. **Lemma 2.38.** If $A^\bullet \to B^\bullet$ is a qis. with $A^\bullet, B^\bullet \in K^+(\mathcal{A})$. Then for any bounded below $I^\bullet$ we have the bijection

$$\mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet).$$

Proof. The idea is we complete $B^\bullet \to A^\bullet$ to a distinguished triangle in $K^+(\mathcal{A})$ and then apply $\mathrm{Hom}(-, I^\bullet)$ to this sequence. We can conclude if $\mathrm{Hom}_{K(\mathcal{A}}(C^\bullet, I^\bullet) = 0$ when $C^\bullet$ is acyclic. We will show this. □

Lemma 2.14. *If $C^\bullet$ is acyclic over $\mathcal{A}$, and if we have $P^\bullet$ a bounded above or dually, $I^\bullet$ bounded below complex, then* $\mathrm{Hom}_K(P^\bullet, C^\bullet) = 0$ *and dually,* $\mathrm{Hom}_K(C^\bullet, I^\bullet) = 0$.

Proof. Suppose we have some $f \in \mathrm{Hom}(P^\bullet, C^\bullet)$ we show it is homotopic to the 0. Then we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{-1} \arrow[r, "d^{-1}"] & C^0 \arrow[r, "d^0"] & C^1 \arrow[r] & \cdots
\end{tikzcd}$$

This means that f_0 factors through $\ker d^0 = \mathrm{img}\, d^{-1}$. So we have the following diagram along with the lift h^0 since P^0 is projective.

$$\begin{tikzcd}
 & P^0 \arrow[dl, dotted, "h^0"'] \arrow[d, "f_0"] \\
C^{-1} \arrow[r, two heads] & \ker d^0
\end{tikzcd}$$

Let h be the chain homotopy that is h^0 in degree 0 and 0 elsewhere, so we replace the orignal f with the homotopic map $\tilde{f} = f - dh + hd$. Note now that $\tilde{f}\colon P^0 \to C^0$ this map is equivalently 0, $\tilde{f_0} \equiv 0$. Proceeding in this way, we can continue to form a homotopy between this map so that each succesive map $P^i \to C^i \equiv 0$, and so we have that $f \sim 0$. $\square$

2.11. **Lemma 2.39 and Proposition 2.40.** Let $A^\bullet, I^\bullet \in Kom^+(\mathcal{A})$. Then

$$Hom_{K(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

The proof of this follows from Lemma 2.38. Since we have to show that any morphism in $D(\mathcal{A})$ of this form,

$$\begin{tikzcd} & B \arrow[dl, "qis"'] \arrow[dr] & \\ A & & I \end{tikzcd}$$

can be completed to the following diagram up to homotopy.

$$\begin{tikzcd} & B \arrow[dl, "qis"'] \arrow[dr] & \\ A \arrow[rr, dotted] & & I \end{tikzcd}$$

Put another way, we have to show that for any qis $B \to A$, we have a bijection $\mathrm{Hom}_K(A, I) \to \mathrm{Hom}_K(B, I)$, which is exactly what the previous lemma gives us.

With these two lemmas, Prop 2.40 follows. Just note that the subtlety they mention is the fact that, you will need to truncate the complex $C^\bullet$ for it to be bounded below.

3. Derived Functor

3.1. **Exact Functors.** If we have an additivie functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ this can be viewed as a functor $\mathcal{F}\colon K(\mathcal{A}) \to K(\mathcal{B})$ by apply $\mathcal{F}$ termwise. Does it descend to the derived category??? If $\mathcal{F}$ is exact, then $\mathcal{F}$ sends qis. to qis. This can be seen for example by the FHHF Theorem [Vak17], which gives a natural isomorphism $H^n F(A^\bullet) \cong F(H^n A^\bullet)$.

3.2. **Definition.** Ok so an exact functor $\mathcal{F}$, can descend to a functor $\mathcal{F}\colon D(\mathcal{A}) \to D(\mathcal{B})$ but most of the functors we care about are not exact, a lot are left exact, Γ, sheaf hom (Also does anyone know how to latex sheaf hom? when I use math cal i get $\mathcal{Hom}$

So assume that $\mathcal{F}$ is a left exact functor, and $\mathcal{A}$ has enough injectives. Let $I_{\mathcal{A}}$ denote the full subcategory of all injectives in $\mathcal{A}$. Then we define the right derived functor as

$$\begin{tikzcd} R\mathcal{F}\colon D^+(\mathcal{A}) \arrow[r] \arrow[d, dotted, shift left] & D^+(\mathcal{B}) \\ K^+(I_{\mathcal{A}}) \arrow[u, "\cong"] \arrow[r, "\mathcal{F}"'] & K^+(B) \arrow[u] \end{tikzcd}$$

3.3. **What is this Explicitly?** Start with some $A^\bullet$ bounded below complex. Then choose a qis. $q\colon A^\bullet \to I^\bullet$ where all the I^n are injectives. Then $R\mathcal{F}(A^\bullet) := F(I^\bullet)$.

3.4. **Higher Derived Functors.** Then we simply define the higher derived functors using cohomology. $R^i\mathcal{F}(A) := H^i R\mathcal{F}(A)$. So this goes back to our computing Ext^i using Hom.

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