Strange Duality for Abelian Surfaces

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Dedication

To my brother.

Acknowledgements

First I would like to thank my mom and dad for their support and encouragement. Not just during my Ph.D. but throughout my whole life. I would like to thank my brother for always making what I first thought impossible seem possible. I would like to thank my friends and loved ones who have been there for me over the years and become a part of my life. I would also like to thank my friends from the department who were embarking on this same journey with me. Having someone else on the same path made it easier to push onwards. Last but not least I would like to thank my advisor Alina for her guidance and advice over the years to help me get to where I am now mathematically.

Abstract of Dissertation

We study a problem on the moduli theory of stable sheaves on surfaces, termed strange duality. Strange duality is a symmetry involving two moduli spaces of stable sheaves and a naturally defined Brill-Noether divisor in their product. Given a pair of moduli spaces of semistable sheaves with fixed topological invariants, one can construct a pair of canonical Theta line bundles over both moduli spaces. Strange duality is a conjectured isomorphism between the global sections of the Theta bundles. The question of strange duality was originally formulated in the setting of stable sheaves on curves, which is also called rank level duality. In the case of surfaces, Le Potier originally formulated strange duality for sheaves on the projective plane. The isomorphism has previously been shown to hold for sheaves with fiber degree 1 on abelian surfaces which are the product of two elliptic curves. This can be used to show it holds for generic abelian surfaces. In this thesis we establish the isomorphism for sheaves from a larger class of ranks and fiber degrees. We also establish partial results in the direction of proving strange duality for all abelian surfaces.

Contents

I. Introduction

1 History of strange duality

In this thesis we aim to study a problem in algebraic geometry, specifically in the moduli theory of stable sheaves, termed *strange duality.* Strange duality is a symmetry involving two moduli spaces of stable sheaves and a naturally defined Brill-Noether divisor in their product. The duality first arose in the string theory literature in the setting of moduli spaces of stable sheaves on curves, and was formulated mathematically in [Bea2] and [DT]. For moduli spaces of stables sheaves on surfaces, the strange duality problem was first posed by Le Potier [LP2] in the setting of $\mathbb{P}^2$ and rational surfaces, and initial progress on the problem came from Le Potier's school, most notably through Danila's work [D1, D2] and Scala's [S].

Let us review very briefly strange duality in the curve setting, which is by now solved. Let C be a smooth projective curve of genus $g \geq 1$ over $\mathbb{C}$. Let $U(k, d)$ be the moduli space of semistable bundles of rank k and degree d on the curve C. To a vector bundle M on C with slope $\mu = g - 1 - d/k$ one can attach a line bundle $\Theta_{k,M}$ on $U(k, d)$ as well as an associated section which on the stable part of the moduli space vanishes on the jumping locus

$$\Theta_{k,M} = \{E \in U(k, d) : h^0(E \otimes M) = h^1(E \otimes M) \neq 0\}.$$

For a generic choice of vector bundle M on C subject to the slope condition, this vanishing locus is a divisor on $U(k, d)$ [RT]. In particular, note that for the moduli space $U(k, k(g - 1))$, there is a canonical divisor (and associated line bundle) supported on the locus

$$\Theta_k = \{E \in U(k, k(g - 1)) : h^0(E) = h^1(E) \neq 0\}. \tag{1}$$

The divisor Θ_k corresponds to choosing $M = \mathcal{O}$. Furthermore, if we let $SU(k, \Lambda)$ be the moduli space of semistable bundles with fixed determinant Λ, then the construction above also produces a Theta line bundle on the moduli space $SU(k, \Lambda)$, which can be shown to depend only on the rank of M. If we let M have the minimal possible rank, the associated line bundle, which we call now θ_k, is ample and generates $\mathrm{Pic}(SU(k, \Lambda))$

[DN]. Under the tensor product map

$$\tau\colon SU(r,\mathcal{O})\times U(k,k(g-1))\to U(kr,kr(g-1)),\quad (E,F)\mapsto E\otimes F$$

one can show the following splitting of line bundles,

$$\tau^*\Theta_{kr}\cong\theta_r^k\boxtimes\Theta_k^r.$$

Note that we call the global sections of $\Theta_k^l=\Theta_k^{\otimes l}$ the generalized theta functions of rank k and level l. The canonical section (1) of Θ_{kr} defines via pullback a section in

$$\mathrm{H}^0(SU(r,\mathcal{O}),\theta_r^k)\otimes\mathrm{H}^0(U(k,k(g-1)),\Theta_k^r).$$

This section induces the strange duality map which exchanges the rank and level of the theta functions,

$$\mathsf{SD}\colon\ \mathrm{H}^0(SU(r,\mathcal{O}),\theta_r^k)^\vee\to\mathrm{H}^0(U(k,k(g-1)),\Theta_k^r).$$

The strange duality problem is to examine whether this map is an isomorphism. This was conjectured in [Bea2] and [DT]. The conjecture was proven for a generic curve in [Bel1, Theorem 1.1], and for all curves in [MO1, Theorem 2].

Beyond the curve setting, strange duality has been studied to great success for surfaces, through the initial work of Le Potier's school in the setting of a rational surface, and later for K3 and abelian surfaces. The tool which proved crucial in obtaining results for K3 and abelian surfaces is the Fourier-Mukai transform on elliptic surfaces first studied in [B]. The idea arose in [MO4] for moduli spaces of stable sheaves over elliptic K3 surfaces. One can use the Fourier-Mukai transform to construct a birational map between moduli spaces which exchanges the ranks and elliptic fiber degrees of their constituent sheaves. Starting with a pair of moduli spaces parametrizing stable sheaves of arbitrary rank but fiber degree one, the transform will produce two moduli spaces parametrizing sheaves of rank one (and some higher fiber degree). In the rank-one setting, strange duality is known to hold [MO2, Proposition 1].

The Fourier-Mukai transform was also used to study strange duality for moduli spaces of stable sheaves over abelian surfaces in [MO5], [BMOY]. These papers notably establish the isomorphism for abelian surfaces $X=B\times F$ which are products of two elliptic curves, and moduli spaces of sheaves over them where each sheaf V satisfies $c_1(V) \cdot f = 1$. From this case, the papers [MO5], [BMOY] deduce that strange duality holds for quite general pairs of moduli spaces over a *generic* abelian surface.

The most recent strange duality push in the K-trivial setting was by Makarova [Mak], who constructed the relative moduli space of stable sheaves over quasipolarized projective surfaces. This allowed Makarova to extend the definition of the relative strange duality morphism to quasipolarized K3 surfaces of degree 2. The morphism was previously only defined for K3 surfaces of degree at least 4 [MO6].

In this thesis we study and extend the known results of strange duality between moduli spaces of stable sheaves on abelian surfaces. We build on the results from [BMOY] by using the Fourier-Mukai transform on elliptic abelian surfaces with higher-rank kernels. We establish the strange duality isomorphism for a more general class of Mukai vector pairs. The main results are stated precisely in the following section.

2 Main results

Given (X, H) a polarized abelian surface over $\mathbb{C}$, and a coherent sheaf $V \to X$, we denote by

$$v = \operatorname{ch} V \sqrt{\operatorname{td} X} \in \mathrm{H}^{2*}(X, \mathbb{Z})$$

its Mukai vector. Given a pair of Mukai vectors

$$v = (v_0, v_2, v_4), \quad w = (w_0, w_2, w_4) \in \mathrm{H}^{2*}(X)$$

their Mukai pairing is given by

$$\langle v, w \rangle = \int_X v_2 w_2 - v_0 w_4 - v_4 w_0.$$

We let M_v denote the moduli space of Gieseker H-semistable sheaves with Mukai vector v. If v is primitive and H is generic, then all semistable sheaves are stable. Hence M_v is a smooth projective variety [HL, Theorem 4.5.4] of dimension

$$\dim \mathsf{M}_v = 2d_v + 2, \quad d_v = \frac{1}{2}\langle v, v \rangle.$$

We let

$$\mathbf{R}\,\mathcal{S} \colon D(X) \to D(\widehat{X})$$

denote the classic Fourier-Mukai transform with respect to the normalized Poincaré bundle on $X \times \widehat{X}$. After fixing a reference sheaf Λ with Mukai vector v, Yoshioka proved [Y, Theorem 4.1] that the following morphism is the Albanese map of M_v,

$$\mathsf{alb}\colon \mathsf{M}_v \to X \times \widehat{X}, \quad V \mapsto (\det \mathbf{R}\,\mathcal{S}(V) \otimes \det \mathbf{R}\,\mathcal{S}(\Lambda)^\vee, \det V \otimes \det \Lambda^\vee)\,.$$

We define a second moduli space K_v as the fiber of the Albanese map over the origin $(\mathit{0}, \mathit{0}) \in X \times \widehat{X}$. The moduli space K_v has dimension $2d_v - 2$. Yoshioka proved that it is deformation equivalent to the generalized Kummer variety $K^{[d_v]}$, hence is irreducible holomorphic symplectic [Y, Theorem 0.2]. Here we recall that if $\mathsf{a}\colon X^{[d_v]} \to X$ denotes the addition map on the abelian variety X, the generalized Kummer variety is defined as

$$K^{[d_v]} = \{Z \in X^{[d_v]} : \mathsf{a}(Z) = 0\} \subset X^{[d_v]}.$$

We now let v and w be a pair of Mukai vectors that are orthogonal in the sense that

$$\langle v^\vee, w\rangle = -\chi(v.w) = 0.$$

Given a pair of reference sheaves $V \in \mathsf{K}_v$ and $W \in \mathsf{M}_w$, we can construct determinant line bundles

$$\Theta_w \to \mathsf{K}_v, \quad \Theta_v \to \mathsf{M}_w$$

following [L] and [LP1], which do not depend on the choices of reference sheaves $V \in \mathsf{K}_v$ and $W \in \mathsf{M}_w$. Then the line bundle associated with the Brill-Noether divisor

$$\Theta = \left\{(V, W) : \mathrm{H}^0(V \overset{\mathbf{L}}{\otimes} W) \neq 0\right\} \subset \mathsf{K}_v \times \mathsf{M}_w \tag{2}$$

satisfies

$$\mathcal{O}\,(\Theta) \cong \Theta_w \boxtimes \Theta_v \text{ on } \mathsf{K}_v \times \mathsf{M}_w.$$

The section (2) can therefore be viewed as an element in

$$\mathrm{H}^0(\mathsf{K}_v, \Theta_w) \otimes \mathrm{H}^0(\mathsf{M}_w, \Theta_v)$$

inducing thus a map, termed *the strange duality morphism*

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v). \tag{3}$$

The question of interest is when this map is an isomorphism. This was established [BMOY, Theorem 1.2] when X is an elliptic abelian surface for pairs of Mukai vectors with elliptic fiber degree 1, and with stability of sheaves being defined with respect to a polarization suitable in the sense of Friedman [F2]. Importantly, as analyzed in [BMOY, §5], under weak assumptions on the Mukai vectors v and w, the question of whether the map (3) is an isomorphism does not depend on the choice of polarization on X. The result can be further used to show that strange duality holds for a generic abelian surface [BMOY, Theorem 1.3].

The ultimate goal in the study of strange duality for abelian surfaces is to establish the isomorphism for all abelian surfaces, not just a generic surface in moduli. Results on the density of abelian surfaces containing elliptic curves in the moduli space of polarized abelian surfaces give a possible avenue to solving global strange duality. We give more details in Chapter VI. To fully make use of such density arguments, it is necessary to have a larger class of pairs of Mukai vectors on an elliptic abelian surface for which the strange duality isomorphism is known to hold. In particular it is important to have examples of strange duality for higher rank and fiber degree Mukai pairs.

This brings us to the main results of the thesis which establish new cases of strange duality on abelian surfaces $X = B \times F$ where B and F are elliptic curves. Throughout, stability of sheaves is defined relative to a suitable polarization in the sense of Friedman [F2]. We have:

Theorem 1. *Let $X = B \times F$ be a product of elliptic curves, viewed as an elliptic fibration over B. Let b and n be nonzero coprime integers. Choose a pair of orthogonal Mukai vectors v and w satisfying the equations*

$$n \operatorname{rk} v - b \operatorname{fdeg} v = 1, \tag{4}$$

$$n \operatorname{rk} w + b \operatorname{fdeg} w = 1. \tag{5}$$

If the following inequality is satisfied

$$\left| \frac{\operatorname{rk} v - \operatorname{rk} w}{b} \right| \geq 4, \tag{6}$$

then the locus

$$\Theta = \left\{ (V, W) : \mathrm{H}^1(V \overset{\boldsymbol{L}}{\otimes} W) \neq 0 \right\} \subset \mathsf{K}_v \times \mathsf{M}_w \tag{7}$$

is a Cartier divisor inducing an isomorphism

$$\mathsf{SD}\colon \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v).$$

This theorem establishes strange duality for a pair of Mukai vectors with coprime ranks and fiber degrees, which satisfy the same linear Diophantine equation up to dualizing v. We obtain this result by starting with a pair of Mukai vectors where strange duality is known to hold, and using a Fourier-Mukai kernel of higher rank (in contrast to the rank-one kernel employed in [BMOY]). By making further numerical choices for the higher-rank Fourier-Mukai kernels, we obtain the following specialization of Theorem 1.

Theorem 2. *Let $X = B \times F$ be a product of elliptic curves. Fix the data of two orthogonal Mukai vectors of the form*

$$\widetilde{v} = (r, \sigma + \delta f, k), \quad \widetilde{w} = (r', \sigma + \delta' f, k')$$

with ranks $r, r' \geq 2$. The orthogonality condition is given by the equation

$$rk' + r'k + \delta + \delta' = 0.$$

We require that

$$\langle \widetilde{v}, \widetilde{v} \rangle \geq 6, \quad \langle \widetilde{w}, \widetilde{w} \rangle \geq 0,$$

which is equivalent to the following inequalities being satisfied

$$\delta \geq rk + 2, \quad \delta' \geq r'k'. \tag{8}$$

Choose a pair of positive coprime integers (x, y) as well as a pair of integers (α, β) such that

$$\alpha x + \beta y = 1.$$

Given these data, for any $t \in \mathbb{Z}$, consider the Mukai vectors

$$v = \Big(y(r + r') + x, \Big((r + r')(\alpha + yt) - (\beta - xt) \Big) \sigma - \Big(k(yr' + x) + \delta y \Big) f, \\ k(\beta - xt) - (kr' + \delta)(\alpha + yt) \Big) \tag{9}$$

$$w = \Big(x, \Big(\beta - xt \Big) \sigma + \Big(\delta' y - k'(yr' + x) \Big) f, (k'r' - \delta')(\alpha + yt) - k'(\beta - xt) \Big). \tag{10}$$

Then the locus

$$\Theta = \left\{ (V, W) : \mathrm{H}^1(V \overset{\boldsymbol{L}}{\otimes} W) \neq 0 \right\} \subset \mathsf{K}_v \times \mathsf{M}_w$$

is a Cartier divisor inducing an isomorphism

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v).$$

The following is a corollary of Theorem 2. It establishes strange duality in a pair consisting of a rank 1 Mukai vector and an odd-rank Mukai vector – a case which had not been previously investigated. Note that these vectors are not limited to having fiber degree 1.

Corollary 1. *Suppose that $R \geq 5$ is an odd integer, and let $D = Rs - 1$ for any $s \in \mathbb{Z}$. Suppose that*

$$k, k' < 0$$

and consider the following pair of Mukai vectors,

$$v = \Big(R, D\sigma - kf, k(1-s) \Big) \tag{11}$$

$$w = \Big(1, (1-s)\sigma - k'Rf, k'D \Big). \tag{12}$$

The map

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v)$$

is an isomorphism.

3 Organization of chapters

In Chapter II we review the main techniques that will be used throughout the thesis. This includes basic facts on abelian varieties, moduli theory of sheaves on surfaces, as well as the Fourier-Mukai transform. We also define the strange duality morphism in this chapter, and the specifics of the setup for abelian surfaces.

In Chapter III we survey the known results on strange duality in the setting of abelian surfaces. This includes the rank-one strange duality theorem on an arbitrary abelian surface, and the fiber degree one strange duality theorem on abelian surfaces which are products of two elliptic curves.

In Chapter IV we present the two new theorems generalizing the previously known cases of strange duality for product abelian surfaces. We also present further interesting new instances of strange duality.

In Chapter V we explain how to use the strange duality setting on product surfaces to derive results for the more general case of nonsplit elliptic abelian surfaces.

Finally in Chapter VI we discuss the open problems that came up in the thesis and also explain that the results of the thesis represent steps toward establishing strange duality for every abelian surface.

II. Background

In this chapter we will survey the foundational objects and results that will be used throughout the thesis. As mentioned in the introduction, the Fourier-Mukai transform plays a central role in proving the main theorems, and so we begin by recalling key results on the derived category of coherent sheaves. We also define the Fourier-Mukai transform in this section. Importantly in this section we also discuss the Grothendieck-Riemann-Roch Theorem which is used throughout the text.

In the next section we introduce the key techniques involving abelian varieties. In particular, we discuss Mukai's original construction of the Fourier-Mukai transform, as well as the Poincaré Complete Reducibility Theorem which is used extensively in Chapter V to reduce questions about elliptic abelian surfaces, to product abelian surfaces. We discuss stability of sheaves, as well as properties of the moduli space of stable sheaves.

With these key geometric objects defined, we explicitly construct the strange duality morphism between moduli spaces of sheaves on abelian surfaces. We conclude the chapter by surveying the key results of Bridgeland on the relative Fourier-Mukai transform defined for elliptic surfaces.

1 Derived category of coherent sheaves

In this section we let X be a scheme over an algebraically closed field. The derived category of X, denoted $D(X)$, is the bounded derived category of $\mathrm{Coh}(X)$, the abelian category of coherent sheaves on X. Usually the category of coherent sheaves will not have enough injectives. So at times we may have to enlarge our category to that of quasicoherent sheaves, which does have enough injectives. We may then carry out our calculations in this larger category. We clarify this idea in the following two propositions.

Proposition 1 ([Huy, Proposition 3.3]). *If X is a Noetherian scheme, and if $\mathcal{F}$ is quasicoherent, then it admits a resolution*

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

by quasicoherent injective $\mathcal{O}_X$-modules.

Proposition 2 ([Huy, Proposition 3.5]). *Let X be a Noetherian scheme. Then the natural functor*

$$D(X) \to D^b(\mathrm{Qcoh}(X))$$

defines an equivalence of categories between $D(X)$ and $D_{coh}(\mathrm{Qcoh}(X))$. Which is the full triangulated subcategory of bounded complexes with coherent cohomology.

If A is an object in the abelian category $\mathcal{A}$, then we may define a left exact functor

$$\mathrm{Hom}(A,-)\colon \mathcal{A} \to \mathbf{Ab}.$$

If this category has enough injectives, then we can define its right derived functor $\mathbf{R}\,\mathrm{Hom}(A,-)$. We define the Ext groups as the higher derived functors of Hom,

$$\mathrm{Ext}^i(A,-) = \mathbf{R}^i\,\mathrm{Hom}(A,-) = \mathrm{H}^i\,\mathbf{R}\,\mathrm{Hom}(A,-). \tag{13}$$

The following proposition details how to reinterpret the previous Ext groups, which provide information about extensions in $\mathcal{A}$, as morphisms in $D(\mathcal{A})$.

Proposition 3. *Suppose $A, B \in \mathcal{A}$ are objects of an abelian category containing enough injectives. Then there are natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]),$$

where we view A and B as complexes concentrated in degree zero.

This result can be generalized to arrive at the following isomorphism, now for complexes

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]). \tag{14}$$

With this in mind we can rephrase Serre duality in the context of the derived category.

Theorem 3 (Serre Duality)**.** *Let X be a smooth projective variety of dimension n, over a field k. Then for any two $A, B \in D(X)$, there exists a functorial isomorphism*

$$\mathrm{Hom}(A, B \otimes \omega_X[n]) \cong \mathrm{Hom}(B, A)^*.$$

We state the theorem of Grothendieck Duality, at least the version that we will use to aide in calculations throughout the thesis. Note that there are more general formulations.

Definition 1. *Let $f\colon X \to Y$ be a morphism of smooth schemes over a field k of relative dimension $\dim(f) = \dim X - \dim Y$. We define the **relative dualizing bundle** as,*

$$\omega_f = \omega_X \otimes f^* \omega_Y^*.$$

Theorem 4 (Grothendieck Duality). *For any $A \in D(X)$ and $B \in D(Y)$, note that these are both the bounded derived categories, there exists a functorial isomorphism*

$$\mathbf{R}\, f_* \, \mathbf{R}\, \mathcal{H}\mathrm{om}\left(A, \boldsymbol{L}f^*(B) \otimes \omega_f[\dim f]\right) \cong \mathbf{R}\, \mathcal{H}\mathrm{om}\left(\mathbf{R}\, f_* A, B\right).$$

Grothendieck duality is a relative version of Serre duality. We can use it to recover the classic formulation, as well as the derived version given in Theorem 3. Let

$$f\colon X \to \mathrm{Spec}(k)$$

be the structure map, suppose that $\dim X = n$, and let $A \in \mathrm{Coh}(X)$. A coherent sheaf over a point $\mathrm{Spec}(k)$ is a finite dimensional vector space, so we simply take $k \in \mathrm{Coh}(\mathrm{Spec}(k))$. Since pushforward to a point is the same as taking global sections, Grothendieck duality gives us the following isomorphism,

$$\mathbf{R}\,\Gamma \circ \mathbf{R}\,\mathcal{H}\mathrm{om}(A, \omega_X[n]) \cong \mathbf{R}\,\mathcal{H}\mathrm{om}(\mathbf{R}\,\Gamma(A), k). \tag{15}$$

We may simplify the left hand side to $\mathbf{R}\,\mathrm{Hom}(A, \omega_X[n])$, and the right hand side to $\mathbf{R}\,\Gamma(A)^\vee$, where this is the derived dual. We shift both sides of the isomorphism by $[i-n]$ to get

$$\mathbf{R}\,\mathrm{Hom}(A, \omega_X[i]) \cong \mathbf{R}\,\Gamma(A)^\vee[i-n].$$

The right hand side is isomorphic to $\mathrm{H}^{n-i}(X, A)^*$, since dualizing reverses degrees. The left hand side is simply $\mathrm{Ext}^i(A, \omega_X)$, and so we recover the classic version of Serre Duality,

$$\mathrm{Ext}^i(A, \omega_X) \cong \mathrm{H}^{n-i}(X, A)^*. \tag{16}$$

Similarly, to recover Theorem 3, we consider now two objects $A, B \in D(X)$. Note that we have the following adjunction in the derived category

$$\mathbf{R}\operatorname{Hom}(A, B \otimes \omega_X[n]) \cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\,\mathcal{H}\mathrm{om}(B, A), \omega_X[n]).$$

By Grothendieck duality the right hand side is isomorphic to

$$\mathbf{R}\,\mathcal{H}\mathrm{om}(\mathbf{R}\,\Gamma \circ \mathbf{R}\,\mathcal{H}\mathrm{om}(B, A), k).$$

Note that this is the same isomorphism from (15) except we have replaced A with $\mathbf{R}\,\mathcal{H}\mathrm{om}(B, A)$. We may simplify this further to

$$\mathbf{R}\operatorname{Hom}(B, A)^{\vee}.$$

Taking the zeroth cohomology of both sides yields the isomorphism

$$\operatorname{Hom}(A, B \otimes \omega_X[n]) \cong \operatorname{Hom}(B, A)^*,$$

which recovers the derived version of Serre duality.

We conclude by defining the Fourier-Mukai transform. Let X and Y be smooth projective varieties and let

$$q \colon X \times Y \to X, \quad p \colon X \times Y \to Y$$

denote the two projections.

Definition 2. *Pick an object $\mathcal{P} \in D(X \times Y)$, which we call the **kernel** of the transform. Then the induced **Fourier-Mukai transform** is the functor*

$$\Phi_{\mathcal{P}} \colon D(X) \to D(Y), \quad E \mapsto \mathbf{R}\,p_* \left(\mathbf{L}q^*(E) \overset{\boldsymbol{L}}{\otimes} \mathcal{P} \right).$$

Note that the projection q is flat, so the derived pullback is just the usual pullback. Similarly, if $\mathcal{P}$ is a complex of locally free sheaves, then the derived tensor product is the usual tensor product. For most calculations in this text, the kernel will be a vector bundle, hence we will be in the situation of the usual tensor product. This section has been very brief, for more details please consult the text by Huybrechts [Huy] for a more complete treatment.

2 Abelian varieties

We begin by collecting some useful facts and definitions about abelian varieties. Let X be an abelian variety of dimension g, and let $\widehat{X}$ be its dual abelian variety. There exists a Poincaré bundle $\mathcal{P} \to X \times \widehat{X}$, normalized such that

$$\mathcal{P}\big|_{0\times \widehat{X}}, \quad \mathcal{P}\big|_{X\times 0}$$

are both trivial. For a point $\widehat{x} \in \widehat{X}$, we have the following isomorphism

$$\mathcal{P}\big|_{X\times\{\widehat{x}\}} \cong \widehat{x}.$$

Letting q and p denote the projections to X and $\widehat{X}$ respectively, we can define the Fourier-Mukai transforms

$$\mathbf{R}\,\mathcal{S} : D(X) \to D(\widehat{X}), \quad \mathbf{R}\,\mathcal{S}(-) = \mathbf{R}\,p_!\left(\mathcal{P} \overset{\mathbf{L}}{\otimes} q^*(-)\right),$$

$$\mathbf{R}\,\widehat{\mathcal{S}} : D(\widehat{X}) \to D(X), \quad \mathbf{R}\,\widehat{\mathcal{S}}(-) = \mathbf{R}\,q_!\left(\mathcal{P} \overset{\mathbf{L}}{\otimes} p^*(-)\right).$$

In Mukai's original paper where he first defined this functor, he proves that these functors define an equivalence of derived categories.

Theorem 5 ([Muk1, Theorem 2.2])**.** *There is an isomorphism of functors*

$$\mathbf{R}\,\widehat{\mathcal{S}} \circ \mathbf{R}\,\mathcal{S} \cong (-\operatorname{id}_X)^*[-g].$$

In particular, the functor $\mathbf{R}\,\mathcal{S}$ induces an equivalence of categories between $D(X)$ and $D(\widehat{X})$.

Since the typical object in $D(X)$ is a complex of sheaves, and for geometry we are primarily interested in studying sheaves on X, we have the following definition.

Definition 3. *We call an object $A \in D(X)$ a* ***sheaf*** *if $\mathrm{H}^i(A) = 0$ for all $i \neq 0$. In other words, if the complex is concentrated in degree zero.*

Note that this is just an instance of the fact that if $\mathcal{A}$ is an abelian category then it embeds into its derived category $D(\mathcal{A})$.

Definition 4 ([Muk1, Definition 2.3])**.** *Given a sheaf V on X, we say that it is* ***WIT*** *if* $\mathbf{R}^i\,\mathcal{S}(V)=0$ *for all but one i. This i is denoted by $i(V)$ and is called the* ***index*** *of V. In other words, V is WIT of index i if*

$$\mathbf{R}\,\mathcal{S}(V)[i]$$

is a sheaf. If V is WIT-i, we denote the coherent sheaf $\mathbf{R}^i\,\mathcal{S}(V)$ on $\widehat{X}$ by $\widehat{V}$ and call it the ***Fourier transform*** *of V.*

Recall that we identify a coherent sheaf with the complex concentrated in degree 0. So if V is WIT-i then $\mathbf{R}\,\mathcal{S}(V)=\widehat{V}[-i]$. Using the previous results, Mukai establishes the following proposition, which is called the Parseval identity.

Proposition 4 ([Muk1, Corollary 2.5])**.** *Let $A, B\in D(X)$ and assume they are WIT of index a and b respectively. Then for every integer n we have the isomorphism,*

$$\operatorname{Ext}^n(A,B)\cong\operatorname{Ext}^{n+a-b}\left(\widehat{A},\widehat{B}\right).$$

Proof.

$$\begin{aligned}\operatorname{Ext}^n(A,B)&\cong\operatorname{Hom}_{D(X)}(A,B[n])\\&\cong\operatorname{Hom}_{D(\widehat{X})}(\mathbf{R}\,\mathcal{S}(A),\mathbf{R}\,\mathcal{S}(B[n]))\\&\cong\operatorname{Hom}_{D(\widehat{X})}\left(\widehat{A}[-a],\widehat{B}[n-b]\right)\\&\cong\operatorname{Ext}^{n+a-b}\left(\widehat{A},\widehat{B}\right).\end{aligned}$$

□

The kernels used to construct the Fourier-Mukai transform often times arise from universal bundles on some fine moduli space of sheaves. Mukai generalizes some of the properties these universal bundles often have, to define a semihomogeneous bundle.

Definition 5 ([Muk2, §Introduction])**.** *A vector bundle V on an abelian variety X is* ***semihomogeneous*** *if for every $x\in X$, there exists a line bundle L on X such that*

$$t_x^*V\cong V\otimes L,$$

where t_x is translation by x on X.

In the case of an elliptic curve F, the existence of a universal bundle is a classic result. We collect some of the properties of the universal bundle on $F \times F$ in the following two lemmas.

Lemma 1 ([MO5, Lemma 3.1])**.** *Let F be an elliptic curve, and let $a > 0$ and b a pair of coprime integers. Then there exists a vector bundle $\mathcal{U} \to F \times F$ such that:*

(i) the restriction of $\mathcal{U}$ to $F \times \{pt\}$ is stable of rank a and degree b

(ii) the restriction of $\mathcal{U}$ to $\{pt\} \times F$ is stable of rank a and degree c.

Furthermore,

$$c_1(\mathcal{U}_F) = c_1(\mathcal{P}_F) + b[0 \times F] + c[F \times 0]. \tag{17}$$

To get an idea for why such a vector bundle exists, we consider the moduli space $M_F(a, b)$ of rank a degree b bundles on F. By Atiyah's classification theorem [A, Theorem 7], we know that $M_F(a, b) \cong F$. We then let $\mathcal{U} \to F \times M_F(a, b)$ be the universal bundle. This bundle is not unique, but can be normalized so that

$$an + bc = 1$$

for some integer n, and so that

$$\det \mathcal{U} = \mathcal{O}\left(b[0 \times F] + c[F \times 0]\right) \otimes \mathcal{P}_F.$$

Lemma 2 ([MO5, Lemma 3.2])**.** *The sheaf $\mathcal{U} \to F \times F$ is semihomogeneous. In particular,*

$$\operatorname{ch} \mathcal{U} = a \exp\left(\frac{c_1(\mathcal{U})}{a}\right).$$

We briefly sketch how to derive the expression for the Chern character of any semihomogeneous bundle. By [Muk2, Proposition 6.15] any semihomogeneous bundle V admits a finite filtration whose associated graded pieces are stable of the same slope

$$\frac{c_1(V)}{\operatorname{rk} V}.$$

Since the Chern character is additive, by virtue of the filtration, we may assume that our bundle V is stable hence simple (Proposition 5). Now by [Muk2, Proposition 7.3] for a simple semihomogeneous bundle V, of $\operatorname{rk} V = r$ on an abelian variety X, there exists an isogeny $\pi\colon Y \to X$, and a line bundle M on Y such that

$$\pi^*(V) = M^{\oplus r}.$$

This gives us the following expression for the first Chern class,

$$c_1(\pi^*(V)) = rc_1(M). \tag{18}$$

By the push-pull formula, we have that

$$\pi_* \operatorname{ch} \pi^*(V) = \pi_* \pi^* \operatorname{ch} V = d \operatorname{ch} V \tag{19}$$

since the isogeny is étale of some degree d. Now we calculate $\pi_* \operatorname{ch} \pi^*(V)$ in a different way as follows,

$$\begin{aligned}
\pi_* \operatorname{ch} \pi^*(V) &\cong \pi_* \operatorname{ch} M^{\oplus r} \\
&\cong \pi_* \left(r \exp(c_1(M))\right) \\
&\cong \pi_* \left(r \exp\left(\frac{c_1(\pi^*(V))}{r}\right)\right) \\
&\cong \pi_* \pi^* \left(r \exp\left(\frac{c_1(V)}{r}\right)\right) = dr \exp\left(\frac{c_1(V)}{r}\right).
\end{aligned}$$

The third isomorphism comes from the expression for the first Chern class given in (18). The final equality comes again from the push pull formula. Equating both calculations of the pushforward, we have the equality

$$d \operatorname{ch} V = dr \exp\left(\frac{c_1(V)}{r}\right). \tag{20}$$

Canceling the degree of the isogeny from both sides gives us the formula for the Chern character in Lemma 2.

The following theorem will prove useful to reduce calculations concerning general elliptic abelian surfaces to split abelian surfaces.

Theorem 6 (Poincaré Complete Reducibility Theorem [Mum1, Chapter IV.19 Theorem 1])**.** *Let (X, H) be a polarized abelian variety. Then for any abelian subvariety Y, there exists a unique complementary abelian subvariety Z, associated with the norm map of Y. The addition map $\tau\colon Y \times Z \to X$ is an isogeny.*

Stronger still, if we consider the restrictions $H|_Y$, $H|_Z$ we have [BL, Corollary 5.3.6]:

$$\tau\colon (Y, H|_Y) \times (Z, H|_Z) \to (X, H) \tag{21}$$

is an isogeny of *polarized* abelian varieties. This in turn implies the important numerical relation

$$\deg \tau \cdot \chi(X, H) = \chi(Y, H|_Y) \cdot \chi(Z, H|_Z) \tag{22}$$

We briefly explain the construction given in [Mum1, Chapter IV.19]. Let $\iota\colon Y \to X$ be the inclusion and $\widehat{\iota}\colon \widehat{X} \to \widehat{Y}$ the dual morphism. Since H is ample, the Mumford morphism

$$\phi_H\colon X \to \widehat{X}, \quad x \mapsto t_x^* H \otimes H^{-1}$$

is an isogeny. The restriction $H|_Y$ is an ample line bundle on Y and we note the corresponding Mumford map

$$\phi_{H|_Y}\colon Y \to \widehat{Y}.$$

There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\phi_H"] & \widehat{X} \arrow[d, "\widehat{\iota}"] \\ Y \arrow[u, "\iota"] \arrow[r, "\phi_{H|_Y}"'] & \widehat{Y} \end{tikzcd}$$

The subvariety Z is the connected component of 0 in $\phi_H^{-1}\left(\ker \widehat{i}\right)$. As the diagram commutes, we have

$$Y \cap Z \subset \ker \phi_{H|_Y},$$

hence we get the inequality

$$\deg \tau = \# Y \cap Z \leq \chi(Y, H|_Y)^2. \tag{23}$$

To realize τ as an isogeny of *polarized* varieties, and therefore deduce the equality (22), please see [BL, Chapter 5.3] for more details.

We also make use of the Seesaw principle often. It can be used to prove when a line bundle on an abelian variety is trivial.

Lemma 3 (Seesaw Lemma [Mum1, Chapter II.5 Corollary 6]). *Let X be a complete variety, T any variety and L a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T : L\big|_{X\times t} \text{ is trivial on } X \times t\}$$

is closed in T, and if on $X \times T_1$, $p\colon X \times T_1 \to T_1$ is the projection, then $L|_{X\times T_1} \cong p^*M$ for some line bundle M on T_1.

We can say more, if the above line bundle L is trivial on at least one fiber of $X \times T_1 \to X$, then the line bundle $L|_{X\times T_1}$ is trivial. We also describe a criterion to determine when two line bundles are isomorphic, at least up to a twist.

Corollary 2 ([Huy, Remark 9.5]). *Suppose we are in the setup from the previous lemma, except now we have two line bundles L and L' such that $L|_t \cong L'|_t$ for all the closed points on T. Then*

$$L \cong L' \otimes p^*M$$

for some line bundle $M \to T$.

3 Stability of sheaves

One of the central geometric objects in our study is the moduli space of stable sheaves on an abelian surface. Before we can discuss properties of the moduli space, we talk about the basic definitions and properties of (semi)stable sheaves. The primary reference for both this section and the next one is [HL].

We start by letting X be a projective scheme over a field k, and fix $n = \dim X$. Recall that the Euler characteristic of a coherent sheaf $\mathcal{F}$ on X is defined by

$$\chi(\mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F}). \tag{24}$$

If we fix an ample line bundle H on X, then we define the Hilbert polynomial $P(\mathcal{F}, t)$ of $\mathcal{F}$ with respect to H by $t \mapsto \chi(\mathcal{F}(tH))$. We may write this polynomial uniquely as

$$P(\mathcal{F}, t) = \sum_{i=0}^{\dim \mathcal{F}} \alpha_i(\mathcal{F}) \frac{t^i}{i!}. \tag{25}$$

We note that the $\alpha_i(\mathcal{F})$ are rational.

Definition 6 ([HL, Definition 1.2.3])**.** *Suppose that* $\dim \mathcal{F} = n$*, then the* ***reduced Hilbert polynomial*** $p(\mathcal{F})$ *of* $\mathcal{F}$ *is given by*

$$p(\mathcal{F}) = \frac{P(\mathcal{F}, t)}{\alpha_n(\mathcal{F})}.$$

With the definition of the reduced Hilbert polynomial we are able to define the notion of (semi)stability of a sheaf. Note that this is also called Gieseker (semi)stability.

Definition 7 ([HL, Definition 1.2.4])**.** *A coherent sheaf* $\mathcal{F}$ *of dimension* n *is* ***semistable*** *if* $\mathcal{F}$ *is pure, and for any proper subsheaf* $\mathcal{G} \subset \mathcal{F}$ *one has*

$$p(\mathcal{G}) \leq p(\mathcal{F}).$$

The sheaf $\mathcal{F}$ *is* ***stable*** *if the inequality is replaced by the strict inequality*

$$p(\mathcal{G}) < p(\mathcal{F}).$$

Recall that we say a coherent sheaf $\mathcal{F}$ is pure of dimension d if for all nontrivial coherent subsheaves $\mathcal{G} \subset \mathcal{F}$ we have that $\dim \mathcal{G} = d$. Again note that the definition of stability depends on the choice of the ample line bundle H.

Proposition 5 ([HL, Corollary 1.2.8])**.** *If* $\mathcal{F}$ *is a stable sheaf, and if* $k = \overline{k}$*, then* $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \cong k$*. In other words* $\mathcal{F}$ *is* ***simple****.*

We can also define the notion of Mumford-Takemoto (semi)stability, or μ-(semi)stability. In the case of curves, these two notions of (semi)stability coincide. However for higher dimensional schemes they differ.

Definition 8 ([HL, Definition 1.2.2])**.** *If* $\mathcal{F}$ *is a coherent sheaf with* $\dim \mathcal{F} = n$*, then we define the* ***rank*** *of* $\mathcal{F}$ *as*

$$\mathrm{rk}\, \mathcal{F} = \frac{\alpha_n(\mathcal{F})}{\alpha_n(\mathcal{O}_X)}.$$

We note that if X is integral, which is the case when X is an abelian variety, that there exists an open dense subset $U \subset X$ such that $\mathcal{F}|_U$ is locally free. Then $\mathrm{rk}\, \mathcal{F}$ is the rank of the locally free sheaf $\mathcal{F}|_U$.

Definition 9 ([HL, Definition 1.2.11])**.** *If* $\mathcal{F}$ *is a coherent sheaf with* $\dim \mathcal{F} = n$*, we define the* ***degree*** *of* $\mathcal{F}$ *as*

$$\deg \mathcal{F} = \alpha_{n-1}(\mathcal{F}) - \mathrm{rk}(\mathcal{F})\alpha_{n-1}(\mathcal{O}_X).$$

*We define the **slope** of $\mathcal{F}$ by*

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}}.$$

If X is a smooth projective variety, such as when X is an abelian variety over $\mathbb{C}$, we can calculate this degree using Hirzebruch-Riemann-Roch theorem, which gives us

$$\deg \mathcal{F} = c_1(\mathcal{F}) \cdot H^{n-1}.$$

Definition 10 ([HL, Definition 1.2.12])**.** *We say an n dimensional coherent sheaf $\mathcal{F}$ is μ-**semistable** if $T_{n-2}(\mathcal{F}) = T_{n-1}(\mathcal{F})$ and*

$$\mu(\mathcal{G}) \leq \mu(\mathcal{F})$$

for all subsheaves $\mathcal{G} \subset \mathcal{F}$ with

$$0 < \operatorname{rk} \mathcal{G} < \operatorname{rk} \mathcal{F}.$$

*We say that $\mathcal{F}$ is μ-**stable** if the inequality above is replaced by the strict inequality*

$$\mu(\mathcal{G}) < \mu(\mathcal{F}).$$

Recall that the $T_i(\mathcal{F})$ are part of the torsion filtration of $\mathcal{F}$. Given a coherent sheaf $\mathcal{F}$ with $d = \dim \mathcal{F}$, there is the unique torsion filtration

$$0 \subset T_0(\mathcal{F}) \subset \cdots \subset T_d(\mathcal{F}) = \mathcal{F}, \tag{26}$$

where $T_i(\mathcal{F})$ is the maximal subsheaf of $\mathcal{F}$ of dimension $\leq i$. The assumption in the definition of μ-(semi)stability simply says that any torsion subsheaf of $\mathcal{F}$ has codimension at least 2. As mentioned before these two notions of (semi)stability differ for schemes of dimension > 1, but they are still related.

Lemma 4 ([HL, Lemma 1.2.13])**.** *If $\mathcal{F}$ is a pure coherent sheaf of dimension n, then we have the following chain of implications*

$$\mathcal{F}\ \mu\text{-stable} \implies \mathcal{F}\ \text{stable} \implies \mathcal{F}\ \text{semistable} \implies \mathcal{F}\ \mu\text{−semistable}.$$

Each of these implications is strict. We construct now a locally free sheaf that is μ-semistable but not semistable. Let $X = \mathbb{P}^2$ and fix a closed point $x \in X$ with ideal sheaf $\mathcal{I}$. We have the ideal sheaf exact sequence,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow k(x) \longrightarrow 0. \tag{27}$$

Twisting this exact sequence by $\omega_X = \mathcal{O}(-3)$ yields

$$0 \longrightarrow \mathcal{I}(-3) \longrightarrow \mathcal{O}(-3) \longrightarrow k(x) \longrightarrow 0. \tag{28}$$

We note that $\mathrm{H}^0(\mathcal{O}(-3)) = 0$, so taking cohomology of (28) implies that $\mathrm{H}^0(k(x)) \subset \mathrm{H}^1(\mathcal{I}(-3))$. Since $\mathrm{H}^0(k(x)) \neq 0$, we have that

$$0 \neq \mathrm{H}^1(\mathcal{I}(-3)) \cong \mathrm{Ext}^1(\mathcal{I}(-3), \mathcal{O}(-3))^* \cong \mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_X)^*.$$

The first isomorphism comes from Serre duality 3. Since this Ext group is nonzero there is a nonsplit extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow V \longrightarrow \mathcal{I} \longrightarrow 0. \tag{29}$$

We introduce now a very important result that allows us to construct rank two vector bundles on a surface X by relating them to codimension 2 subschemes.

Theorem 7 (Serre Correspondence [HL, Theorem 5.1.1])**.** *Let X be a surface, and let $Z \subset X$ be a local complete intersection of codimension two. Let L and M be line bundles on X. Then there exists an extension*

$$0 \longrightarrow L \longrightarrow V \longrightarrow M \otimes \mathcal{I}_Z \longrightarrow 0$$

such that V is locally free if and only if the pair

$$(L^\vee \otimes M \otimes \omega_X, Z)$$

has the Cayley-Bacharach property: If $Z' \subset Z$ is a subscheme with $\ell(Z') = \ell(Z) - 1$ and $s \in \mathrm{H}^0(X, L^\vee \otimes M \otimes \omega_X)$ with $s|_{Z'} = 0$, then $s|_Z = 0$.

If we consider the extension (29) then the corresponding line bundles L and M as in the theorem are trivial. Since $\mathrm{H}^0(\omega_X) = 0$, by Theorem 7 the sheaf V is locally free of rank 2. By the exact sequence (29) we have that

$$c_1(V) = c_1(\mathcal{O}_X) + c_1(\mathcal{I}) = 0.$$

We fix the ample line bundle $\mathcal{O}(1) = \mathcal{O}(H)$ for a hyperplane H. With respect to this line bundle the slope $\mu(V) = 0$. Suppose that V is not $\mu-$semistable, since V has rank 2 it would mean there is a line bundle $L \subset V$ such that

$$\mu(L) = c_1(L).H = d > 0.$$

Note that this implies that $\mathrm{H}^0(L^\vee) = 0$. If we consider the sequence (27), twist it by $L^\vee$ and take cohomology we find that

$$\mathrm{H}^0(L^\vee \otimes \mathcal{I}) \subset \mathrm{H}^0(L^\vee) = 0.$$

If we twist the sequence (29) by $L^\vee$ and take cohomology we have the following sequence,

$$0 \longrightarrow \mathrm{H}^0(L^\vee) \longrightarrow \mathrm{H}^0(L^\vee \otimes V) \longrightarrow \mathrm{H}^0(L^\vee \otimes \mathcal{I}) \longrightarrow \cdots$$

Since the two terms surrounding $\mathrm{H}^0(L^\vee \otimes V)$ vanish it implies that the middle term also vanishes. But this contradicts the fact that $L \subset V$, and so

$$0 \neq \mathrm{Hom}(L, V) \cong \mathrm{H}^0(L^\vee \otimes V).$$

We conclude that V is μ-semistable.

Now we show that V is not Gieseker semistable. Since $\omega_X = \mathcal{O}(-3)$ this implies that $c_1(\mathcal{T}_X) = 3H$. From this we calculate that

$$\mathrm{td}(X) = 1 + 3/2H + H^2. \tag{30}$$

We also have the following two expressions,

$$\mathrm{ch}\, V(t) = (2 - H^2)(1 + tH + 1/2t^2H^2) = 2 + 2tH + (t^2 - 1)H^2 \tag{31}$$

$$\mathrm{ch}\, \mathcal{I}(t) = (1 - H^2)(1 + tH + 1/2t^2H^2) = 1 + tH + (1/2t^2 - 1)H^2. \tag{32}$$

With this in mind we use Hirzebruch-Riemann-Roch to calculate the following Hilbert polynomials,

$$P(V) = \int_X \Big(2 + 2tH + (t^2 - 1)H^2\Big)\Big(1 + 3/2H + H^2\Big) = t^2 + 3t + 1 \tag{33}$$

$$P(\mathcal{I}) = \int_X \Big(1 + tH + (1/2t^2 - 1)H^2\Big)\Big(1 + 3/2H + H^2\Big) = 1/2t^2 + 3/2t. \tag{34}$$

Notice that the reduced Hilbert polynomial of $\mathcal{I}$ is $p(\mathcal{I}) = P(\mathcal{I})$, which is less than $p(V) = P(V)/2$. Hence $\mathcal{I}$ is a destabilizing quotient of V.

The following is an example of a stable sheaf that is not μ-stable. This example is due to Maruyama and is presented in [OSS, Chapter 2.1.2], where there are many more interesting examples of vector bundles. Let G be a rank 2 $\mu-$stable bundle on $\mathbb{P}^2$ with

$$c_1(G) = 0, \quad \mathrm{H}^1(\mathbb{P}^2, G) \neq 0.$$

A nonzero element $\xi \in \mathrm{H}^1(\mathbb{P}^2, G)$ defines a nontrivial extension

$$0 \longrightarrow G \longrightarrow V \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0. \tag{35}$$

The bundle V is stable but not μ-stable.

We conclude this section by giving a simple criterion for when a semistable sheaf $\mathcal{F}$ is stable.

Lemma 5 ([HL, Lemma 1.2.14])**.** *Let X be integral. If a coherent sheaf $\mathcal{F}$ with $\dim \mathcal{F} = \dim X$ is μ-semistable, and $\operatorname{rk} \mathcal{F}$ is coprime to $\deg \mathcal{F}$, then $\mathcal{F}$ is μ-stable.*

Proof. Suppose that $\mathcal{F}$ is not μ-stable. Then there exists a subsheaf $\mathcal{G} \subset \mathcal{F}$ with $0 < \operatorname{rk} \mathcal{G} < \operatorname{rk} \mathcal{F}$ such that

$$\deg(\mathcal{F}) \operatorname{rk}(\mathcal{G}) = \deg(\mathcal{G}) \operatorname{rk}(\mathcal{F}).$$

Since $\operatorname{rk} \mathcal{F}$ does not divide $\deg \mathcal{F}$ by the coprimality assumption, this implies that $\operatorname{rk} \mathcal{F} | \operatorname{rk} \mathcal{G}$. This contradicts the fact that $\operatorname{rk} \mathcal{G} < \operatorname{rk} \mathcal{F}$. $\square$

Note that by the chain of implications in Lemma 4 a semistable sheaf $\mathcal{F}$ with coprime rank and degree is stable.

4 Moduli spaces of sheaves

If X is a smooth $K-$trivial surface over $\mathbb{C}$, then for any coherent sheaf $V \to X$, we define its Mukai vector as

$$v(V) = \operatorname{ch} V.\sqrt{\operatorname{td}(X)} \in \mathrm{H}^*(X,\mathbb{Q}).$$

Recall that for two coherent sheaves V, W we can define the Euler characteristic of the pair

$$\chi(V,W) = \sum_i (-1)^i \operatorname{ext}^i(V,W).$$

If X is smooth and projective, then

$$\chi(V,W) = \int_X \operatorname{ch}^\vee(V).\operatorname{ch}(W).\operatorname{td}(X).$$

In this way we can define a bilinear pairing, the Mukai pairing on two Mukai vectors, as

$$\langle v, w\rangle = -\int_X v^\vee.w$$

and it follows that $\chi(V,W) = -\langle v(V), v(W)\rangle$.

Suppose now that (X,H) is a polarized abelian surface, we denote by M_v the moduli space of Gieseker H-semistable sheaves with Mukai vector v. If we pick v to be primitive and H generic, then M_v consists only of stable sheaves and is smooth of dimension

$$\dim \mathsf{M}_v = \langle v, v\rangle + 2. \tag{36}$$

Note that the dimension of the moduli space M_v is given by $\operatorname{rk}\operatorname{Ext}^1(V,V)$, for a general element $V \in \mathsf{M}_v$. We have that

$$\begin{aligned}\dim \mathsf{M}_v &= \operatorname{ext}^1(V,V)\\ &= -\chi(V,V) + \operatorname{ext}^0(V,V) + \operatorname{ext}^2(V,V)\\ &= \langle v, v\rangle + 2,\end{aligned}$$

where $\operatorname{ext}^0(V,V) = \operatorname{ext}^2(V,V) = 1$ by stability of V and Serre duality on X. Please see [HL] for basic properties of these moduli spaces.

Now let $\mathrm{K}_0\left(v^{\perp}\right)$ be the subgroup inside the K-theory of X generated by sheaves with Mukai vectors w orthogonal to v in the sense that

$$\langle v^{\vee}, w\rangle = -\chi(X, v.w) = 0.$$

There is a morphism

$$\Theta\colon K_0(v^{\perp}) \to \mathrm{Pic}(\mathsf{M}_v), \quad [W] \mapsto \Theta_W$$

which was constructed in [LP1] and [L, §1] for surfaces. If M_v is a fine moduli space such that a universal sheaf $\mathcal{V}$ exists on $\mathsf{M}_v \times X$, we define

$$\Theta_W = \det \mathbf{R}\, p_!\left(\mathcal{V} \otimes q^* W\right)^{-1} \to \mathsf{M}_v, \tag{37}$$

where p and q are the projections to M_v and X respectively. If there is no universal sheaf $\mathcal{V}$, we can sill construct a well defined line bundle Θ_W by descent from the Quot scheme. This procedure requires the orthogonality condition.

Now we fix some reference sheaf Λ with Mukai vector v, and use the Fourier-Mukai transform $\mathbf{R}\,\mathcal{S}\colon D(X) \to D(\widehat{X})$, to define the following morphism

$$\mathsf{alb}\colon \mathsf{M}_v \to X \times \widehat{X}, \quad V \mapsto \left(\det \mathbf{R}\,\mathcal{S}(V) \otimes \det \mathbf{R}\,\mathcal{S}(\Lambda)^{\vee}, \det V \otimes \det \Lambda^{\vee}\right).$$

Yoshioka proved that this morphism is the Albanese map of M_v [Y, Theorem 4.1]. We let K_v be the fiber of the Albanese map alb over the origin of $X \times \widehat{X}$. In other words K_v is the moduli space of semistable sheaves with Mukai vector v that have both a fixed determinant and fixed determinant of their Fourier-Mukai transform. Since $\dim \mathsf{M}_v = \langle v, v\rangle + 2$, and the Albanese morphism maps to $X \times \widehat{X}$ which has dimension 4, we have that

$$\dim \mathsf{K}_v = \dim \mathsf{M}_v - 4 = \langle v, v\rangle - 2. \tag{38}$$

These K_v are very special, in fact Yoshioka proves that they are irreducible holomorphic symplectic manifolds [Y, Theorem 0.2], which are also called hyperkähler manifolds. By the Beauville-Bogomolov Decomposition Theorem [Bea1, Theorem 1] every compact Kähler manifold with trivial first Chern class is, up to a finite étale cover, the product of complex tori, Calabi-Yau manifolds, and hyperkähler manifolds.

Despite their significance there are not many examples of such manifolds. This is just one of many indications of the rich geometry surrounding the moduli of sheaves on K3 and abelian surfaces.

Since K_v is a subspace of M_v, we may restrict the line bundle $\Theta_W \to \mathsf{M}_v$ to get a line bundle $\Theta_W \to \mathsf{K}_v$. It is important to consider how these line bundles vary depending on the choice of sheaf W in (37). This is made precise in the following lemma.

Lemma 6 ([MO3, Lemma 1]). *Any sheaf $W \in \mathsf{M}_w$ defines a line bundle Θ_W, as in (37), on the moduli space M_v. The restriction of Θ_W to K_v is independent of the choice of W.*

We briefly sketch why this is the case. Since K_v is irreducible holomorphic symplectic, it is in particular simply connected. So its first integral cohomology vanishes,

$$\mathrm{H}^1(\mathsf{K}_v, \mathbb{Z}) \cong \pi_1(\mathsf{K}_v)^{\mathrm{ab}} = 0.$$

By the Hodge decomposition, $\mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1} = \mathrm{H}^1(\mathsf{K}_v, \mathbb{C}) = 0$. This implies the following vanishing of cohomology

$$\mathrm{H}^1(\mathsf{K}_v, \mathcal{O}_{\mathsf{K}_v}) = 0. \tag{39}$$

We consider the long exact sequence induced by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathsf{K}_v} \xrightarrow{\exp} \mathcal{O}^*_{\mathsf{K}_v} \longrightarrow 0 \tag{40}$$

$$\mathrm{H}^1(\mathsf{K}_v, \mathcal{O}_{\mathsf{K}_v}) \longrightarrow \mathrm{H}^1(\mathsf{K}_v, \mathcal{O}^*_{\mathsf{K}_v}) \xrightarrow{c_1} \mathrm{H}^2(\mathsf{K}_v, \mathbb{Z}) \longrightarrow \cdots \tag{41}$$

Since the term on the left of the exact sequence vanishes, the first Chern class map is an injection of the Picard group into $\mathrm{H}^2(\mathsf{K}_v, \mathbb{Z})$, namely

$$\mathrm{Pic}(\mathsf{K}_v) \xhookrightarrow{c_1} \mathrm{H}^2(\mathsf{K}_v, \mathbb{Z}).$$

Intuitively this means that the Picard group of K_v is discrete. So as long as the topological invariants of a line bundle on K_v are fixed, namely its Mukai vector, then the line bundle is also fixed. This is exactly the situation we are in when we define $\Theta_W \to \mathsf{K}_v$, since it has Mukai vector w. From now on, we can call this line bundle $\Theta_w \to \mathsf{K}_v$ since it only depends on w and not on a choice of $W \in \mathsf{M}_w$.

5 Strange duality morphism

We begin by setting the stage to describe the strange duality morphism. Let (X, H) denote a polarized abelian surface, and fix a pair of orthogonal Mukai vectors v and w.

Then there exists a natural determinant line bundle on the product $\mathsf{M}_v \times \mathsf{M}_w$. We let

$$p \colon \mathsf{M}_v \times \mathsf{M}_w \times X \to \mathsf{M}_v \times \mathsf{M}_w$$

$$q_1 \colon \mathsf{M}_v \times \mathsf{M}_w \times X \to \mathsf{M}_v \times X$$

$$q_2 \colon \mathsf{M}_v \times \mathsf{M}_w \times X \to \mathsf{M}_w \times X$$

denote the natural projections. If there exist universal sheaves

$$\mathcal{V} \to \mathsf{M}_v \times X, \quad \mathcal{W} \to \mathsf{M}_w \times X$$

we may define this line bundle as

$$\Theta = \det \mathbf{R}\, p_! \left(q_1^* \mathcal{V} \otimes q_2^* \mathcal{W} \right)^{-1}. \tag{42}$$

If there are no universal sheaves then we can still construct this line bundle by descent from the Quot scheme. This line bundle always exists and comes with a natural section θ, although it may define a degenerate divisor.

Note that the orthogonality assumption implies that for any $(V, W) \in \mathsf{M}_v \times \mathsf{M}_w$ the Euler characteristic of their product vanishes,

$$\chi(V \otimes W) = 0.$$

We further assume the vanishing of the higher cohomology,

$$\mathrm{H}^2(V \otimes W) = 0. \tag{43}$$

Then the divisor associated to the line bundle Θ is supported on the following jumping locus,

$$\mathcal{O}\left(\Theta\right) = \left\{ (V, W) : \mathrm{H}^0(V \overset{\mathbf{L}}{\otimes} W) \neq 0 \right\} \subset \mathsf{M}_v \times \mathsf{M}_w.$$

Now we restrict the determinant line bundle to $\mathsf{K}_v \times \mathsf{M}_w$, and denote it again by

$$\Theta \to \mathsf{K}_v \times \mathsf{M}_w.$$

Over the product $\mathsf{K}_v \times \mathsf{M}_w$ the line bundle Θ actually splits. This phenomenon does not happen on $\mathsf{M}_v \times \mathsf{M}_w$. For any $W \in \mathsf{M}_w$, if we restrict Θ to the fiber over W, we get

$$\Theta\big|_{\mathsf{K}_v \times W} = \det \mathbf{R}\, p_!(q_1^* \mathcal{V} \otimes W)^{-1} \cong \Theta_w.$$

As was shown in Lemma 6 this line bundle does not vary depending on our choice of sheaf W. Let

$$\pi_1 \colon \mathsf{K}_v \times \mathsf{M}_w \to \mathsf{K}_v, \quad \pi_2 \colon \mathsf{K}_v \times \mathsf{M}_w \to \mathsf{M}_w$$

denote the projection maps. Since the restriction of Θ to every fiber over M_w is constant, by the Seesaw principle 3

$$\Theta \cong \pi_1^* \Theta_w \otimes \pi_2^* L$$

where L is a line bundle on M_w. To determine what L is, we now restrict Θ to a fiber over K_v, and similarly we find that

$$\Theta\big|_{V \times \mathsf{M}_w} = \det \mathbf{R}\, p_!(V \otimes q_2^* \mathcal{W})^{-1} \cong \Theta_V.$$

Again by the Seesaw principle, this line bundle must be constant over each fiber, so we may denote it by Θ_v. So we find that our theta line bundle splits as

$$\Theta \cong \Theta_w \boxtimes \Theta_v. \tag{44}$$

Although at first glance the product $\mathsf{M}_v \times \mathsf{M}_w$ seems to be the natural setting to study strange duality, in fact the asymmetric setting of the product $\mathsf{K}_v \times \mathsf{M}_w$ will be more natural. This is thanks to the splitting of the determinant line bundle, which only occurs on $\mathsf{K}_v \times \mathsf{M}_w$. This splitting is crucial for us to define and study strange duality. By the above splitting, the canonical section of Θ can be viewed as an element in

$$\mathrm{H}^0(\mathsf{K}_v, \Theta_w) \otimes \mathrm{H}^0(\mathsf{M}_w, \Theta_v)$$

which induces the following morphism on cohomology, which we call the strange duality morphism

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_v,\Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w,\Theta_v). \tag{45}$$

The following theorem gives a clue that the global sections of these two determinant line bundles have the same dimension, at least when the higher cohomology vanishes. We let

$$d_v = \frac{1}{2}\dim \mathsf{M}_v, \quad d_2 = \frac{1}{2}\dim \mathsf{M}_w.$$

Theorem 8 ([MO3, Theorem 3])**.** *Assume that the Néron-Severi group of X has rank 1. For the same hypotheses as in* [MO3, Theorem 1]*, we have*

$$\chi(\mathsf{K}_v,\Theta_w) = \chi(\mathsf{M}_w,\Theta_v) = \frac{d_v^2}{d_v+d_w}\binom{d_v+d_w}{d_v}. \tag{46}$$

The fundamental goal of studying strange duality is to answer the following question.

Question 1. *When does this map (45) exist, and when is it an isomorphism?*

6 Fourier-Mukai transform on elliptic surfaces

In this section we explain how to setup a relative version of the Fourier-Mukai transform for an elliptic surface. This was first studied by Bridgeland in [B].

Definition 11. *An elliptic surface is a smooth two dimensional variety X with a relatively minimal morphism*

$$\pi\colon X \to B$$

to a smooth base curve B, such that the general fiber of π is a genus one curve.

Note that by relatively minimal we mean that the fibers of the the projection to B contain no (-1)-curves. For what follows below we will assume that X has irreducible fibers, and that we have picked a polarization that is suitable in the sense of Friedman.

Definition 12 ([F1, Definition 2.1])**.** *A polarization H on an elliptic surface X is suitable, if for all $L \in \mathrm{Pic}(X)$, with $L^2 \geq -c$ for some positive integer c, and $L.f \neq 0$, we have $sign(L.f) = sign(H.L)$.*

Given an object $V \in D(X)$, we define the fiber degree of V to be

$$\operatorname{fdeg} V = c_1(V).f \tag{47}$$

where f is the algebraic equivalence class of a fiber of π. When we considered the Fourier-Mukai transform on an elliptic curve F, we constructed the kernel of the transform as the universal bundle over $F \times M_F(a, b)$. Recall that $M_F(a, b)$ is the moduli space of rank a degree b bundles on F, and furthermore this moduli space is isomorphic to the elliptic curve F. The construction in this section mirrors to some extent the curve setting, except now we consider moduli of fiber sheaves.

Definition 13. *We say a sheaf $V \to X$ is a fiber sheaf if*

$$\operatorname{rk} V = \operatorname{fdeg} V = 0.$$

This is equivalent to V being supported on a finite number of fibers of π.

Let us denote by μ_X the minimal fiber degree of all effective curves in X,

$$\mu_X = \min\{C.f,\ \text{for}\, C \subset X\, \text{effective curve}\}.$$

Take two integers $a > 0$ and b such that $a\mu_X$ and b are coprime. Bridgeland establishes the existence of the fine moduli scheme

$$J_X(a, b)$$

that parametrizes stable fiber sheaves on X of rank a and fiber degree b. It contains an open subset of points which correspond to stable vector bundles of type (a, b) on a nonsingular fiber of π. The scheme $Y = J_X(a, b)$ is also an elliptic surface over the same base $\pi' \colon Y \to B$.

Proposition 6. *There exists a universal sheaf $\mathcal{P}$ on $X \times_B Y$ such that for any point $y \in Y$ the sheaf it corresponds to is $\mathcal{P}_y$. We may push this universal sheaf forward along the map*

$$X \times_B Y \to X \times Y$$

to a sheaf

$$\begin{array}{c}\mathcal{P}\\ \downarrow\\ X\times Y\end{array}$$

on the product.

We also define $\mathcal{Q} = (\mathcal{P}^\vee \otimes \pi^*\omega_X)\,[1]$, and use these two objects in $D(X \times Y)$ to define the following two Fourier-Mukai transforms

$$\Phi(-) = \mathbf{R}\,\pi_{X,*}\,(\mathcal{P} \otimes \pi_Y^*(-))\,, \quad \Psi(-) = \mathbf{R}\,\pi_{Y,*}\,(\mathcal{Q} \otimes \pi_X^*(-))\,.$$

The functor $\Phi[1]$ is a left adjoint to Φ, and so they establish an equivalence between the categories $D(X)$ and $D(Y)$.

Theorem 9 ([B, Theorem 1.2]). *Given integers $a > 0$ and b with $a\delta_X$ coprime to b, where δ_X is the highest common factor of the fiber degrees of sheaves on X, let $Y = J_X(a,b)$. Then there exist tautological sheaves on $X \times Y$, supported on $X \times_B Y$, and for each such sheaf $\mathcal{P}$, the functor $\Phi^{\mathcal{P}}_{Y\to X}\colon D(Y) \to D(X)$ is an equivalence of categories.*

The following lemmas help to establish when the Fourier-Mukai transforms above send a sheaf to a sheaf, at least up to a shift.

Lemma 7 ([B, Lemma 6.1]). *For any sheaf V on X there is a unique short exact sequence*

$$0 \to A \to V \to B \to 0,$$

such that A is WIT-0 with respect to Ψ, and B is WIT-1.

Lemma 8 ([B, Lemma 6.4]). *Let V be a torsion free sheaf on X such that the restriction of V to the general fiber of π is stable. Suppose that $\mu(V) < b/a$. Then V is WIT-1 with respect to Ψ.*

We can specify how the ranks and fiber degrees of a sheaf transform under these Fourier-Mukai transforms. Fix an element

$$\begin{bmatrix} c & a \\ d & b \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

such that δ_X divides d and $a > 0$. Let $Y = J_X(a,b)$, then there exists sheaves $\mathcal{P}$ on $X \times Y$ such that for any point $(x,y) \in X \times Y$, $\mathcal{P}_y$ has Chern class $(0, af, -b)$ on X and $\mathcal{P}_x$ has class $(0, af, -c)$ on Y. Using such a sheaf as the kernel of Φ we have that for any object $V \in D(Y)$,

$$\begin{pmatrix} \operatorname{rk} \Phi(V) \\ \operatorname{fdeg} \Phi(V) \end{pmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{pmatrix} \operatorname{rk} V \\ \operatorname{fdeg} V \end{pmatrix}.$$

Bridgeland establishes how these sheaves transform. As a result, we may pick a universal sheaf $\mathcal{P}$, which amounts to picking an element in $\mathrm{SL}_2(\mathbb{Z})$, such that any sheaf $V \in \mathsf{M}_v$ gets sent to a torsion free sheaf of rank 1 and fiber degree 0, up to a shift. In other words, the sheaf gets sent to

$$I_Z \otimes L[-1].$$

Where I_Z is an ideal sheaf with Z a subscheme of length d_v, and L is a line bundle coming from Y.

Theorem 10 ([B, Theorem 1.1])**.** *Let v be a Mukai vector such that $\operatorname{rk} v = r > 1$ is coprime to $c_1(v).f$. Then we have a birational isomorphism*

$$\mathsf{M}_v \dashrightarrow Y^{[d_v]} \times \widehat{Y},$$

where $Y = J_X(a,b)$ for the unique pair of integers satisfying $br - a(c_1(v).f) = 1$, and $0 < a < r$.

III. Previous results on product surfaces

1 Rank one Mukai vectors

Establishing the strange duality isomorphism for rank 1 Mukai vectors is the first step to study the duality on abelian surfaces. With this in mind we let

$$L \to X$$

be an ample line bundle, and write $\chi(L) = a + b$ for positive integers a and b. Consider the Mukai vectors

$$v = (1, 0, -a), \quad w = (1, c_1(L), a).$$

Then we have the isomorphisms

$$\mathsf{K}_v \cong K^{[a]}, \quad \mathsf{M}_w \cong X^{[b]} \times \widehat{X}.$$

Recall that $K^{[a]}$ is the generalized Kummer variety,

$$K^{[a]} = \{Z \in X^{[a]} : \mathsf{a}(Z) = 0\}.$$

With this setup the following theorem establishes the strange duality isomorphism.

Theorem 11 ([BMOY, Theorem 1.1])**.** *Let $L \to X$ be an ample line bundle on an arbitrary abelian surface. Write $\chi(X, L) = \chi = a + b$ for positive integers a and b. The divisor*

$$\Theta_L = \left\{ (I_Z, I_W, y) : \mathrm{H}^0\left(I_Z \otimes I_W \otimes y \otimes L\right) \neq 0 \right\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}$$

induces an isomorphism

$$\mathsf{SD}_L \colon \mathrm{H}^0\left(K^{[a]}, \Theta_w\right)^\vee \to \mathrm{H}^0\left(X^{[b]} \times \widehat{X}, \Theta_v\right).$$

2 Fiber degree one Mukai vectors

Now, let

$$X = B \times F \to B$$

be a product of elliptic curves, which is an elliptic abelian surface trivially fibered over B. Let f denote the class of the fiber over the origin, and σ the class of the zero section. We consider stability with respect to the polarization

$$H = \sigma + Nf, \quad N \gg 0. \tag{48}$$

Such a polarization is suitable (Definition 12) in the sense of Friedman [F2]. Let $\mathcal{P}_F \to F \times F$ denote the normalized Poincaré bundle on the product. This is the line bundle with first Chern class

$$c_1(\mathcal{P}_F) = \Delta_F - [\mathit{0} \times F] - [F \times \mathit{0}]. \tag{49}$$

The fiber product is the threefold

$$X \times_B X = B \times F \times F.$$

Let $\pi\colon X \times_B X \to F \times F$ denote the projection. In [BMOY], the authors consider the relative Fourier-Mukai transform

$$\mathsf{FM}\colon D(X) \to D(X)$$

with kernel $\pi^*\mathcal{P}_F$. For Mukai vectors v and w with ranks r and r' and fiber degree one, this transform gives rise to birational morphisms

$$\mathsf{K}_v \dashrightarrow K^{[d_v]}, \quad \mathsf{M}_w \dashrightarrow X^{[d_w]} \times \widehat{X}.$$

More concretely, let $V \in \mathsf{K}_v$ and $W \in \mathsf{M}_w$, then

$$\mathsf{FM}(V^\vee) = I_Z(r\sigma - \chi(v)f)[-1], \tag{50}$$

$$\mathsf{FM}(W) = I_{Z'}^\vee \otimes \mathcal{O}\left(-r'\sigma + \chi(w)f\right) \otimes y^{-1}. \tag{51}$$

Here $Z \in K^{[d_v]}$, $Z' \in X^{[d_w]}$, and y is a degree zero line bundle in $\widehat{X}$. Let

$$L = \mathcal{O}\big((r + r')\sigma - (\chi(v) + \chi(w))f\big), \tag{52}$$

and consider the following calculation,

$$\begin{aligned}\mathrm{H}^0(V \overset{\mathbf{L}}{\otimes} W) = \mathrm{Hom}_{D(X)}\left(V^\vee, W\right) &= \mathrm{Hom}_{D(X)}\left(\mathsf{FM}(V^\vee), \mathsf{FM}(W)\right) \\ &= \mathrm{Ext}^1\left(I_Z \otimes y \otimes L, I_{Z'}^\vee\right) \\ &= \mathrm{H}^1\left(I_Z \otimes I_{Z'} \otimes y \otimes L\right)^\vee .\end{aligned}$$

Since the induced birational map

$$\mathsf{K}_v \times \mathsf{M}_w \dashrightarrow K^{[d_v]} \times X^{[d_w]} \times \widehat{X}$$

is regular in codimension 1, we can identify the divisor

$$\Theta = \left\{(V, W) : \mathrm{H}^0(V \overset{\mathbf{L}}{\otimes} W) \neq 0\right\} \subset \mathsf{K}_v \times \mathsf{M}_w$$

with the divisor

$$\Theta_L = \left\{(I_Z, I_{Z'}, y) : \mathrm{H}^0(I_Z \otimes I_{Z'} \otimes y \otimes L) \neq 0\right\} \subset K^{[d_v]} \times X^{[d_w]} \times \widehat{X}.$$

This gives the general idea on how strange duality was extended beyond rank 1 Mukai vectors in the following theorem.

Theorem 12 ([BMOY, Theorem 1.2])**.** *Let $X = B \times F$ be a product abelian surface. Assume that v and w are two orthogonal Mukai vectors of rank $r, r' \geq 2$ with*

$$c_1(v).f = c_1(w).f = 1.$$

Then the locus

$$\Theta = \left\{(V, W) : \mathrm{H}^0\left(V \overset{\boldsymbol{L}}{\otimes} W\right) \neq 0\right\} \subset \mathsf{K}_v \times \mathsf{M}_w$$

is a divisor and induces an isomorphism

$$\mathsf{SD}\colon \mathrm{H}^0\left(\mathsf{K}_v, \Theta_w\right)^\vee \to \mathrm{H}^0\left(\mathsf{M}_w, \Theta_v\right).$$

IV. New results on product surfaces

1 Characterizing the Albanese fiber

Fix a reference sheaf Λ with Mukai vector v. Yoshioka proved that the following morphism is the Albanese map of M_v,

$$\mathsf{alb}\colon \mathsf{M}_v \to X \times \widehat{X}, \quad V \mapsto (\det \mathbf{R}\,\mathcal{S}(V) \otimes \det \mathbf{R}\,\mathcal{S}(\Lambda)^\vee, \det V \otimes \det \Lambda^\vee)\,.$$

Recall that we define K_v as the fiber of the Albanese map over the origin of $X \times \widehat{X}$. So K_v is the moduli space of semistable sheaves with Mukai vector v that have both a fixed determinant and fixed determinant of their Fourier-Mukai transform. In the following lemma we characterize when a sheaf is in K_v, in terms of its second Chern class.

Proposition 7. *Let $X = B \times F$ be the product of elliptic curves, and fix a Mukai vector*

$$v = (r, d\sigma + \delta f, k) \in \mathrm{H}^{2*}(X)$$

We consider a general sheaf $V \in \mathsf{M}_v$, with $c_2(V) = Z \in \mathrm{A}^2(X)$. Then $V \in \mathsf{K}_v$ if and only if $\mathsf{a}(Z) = 0$, where a is the addition map on the abelian surface X.

Proof. We write the first Chern class as an element in Chow,

$$c_1(v) = d\sigma + \delta f \in \mathrm{A}^1(X). \tag{53}$$

Note that the we have the following isomorphism of the fourfold

$$X \times \widehat{X} \cong B \times F \times \widehat{B \times F} \cong B \times F \times \widehat{B} \times \widehat{F}.$$

Since elliptic curves are isomorphic to their duals, we conclude that

$$X \times \widehat{X} \cong B \times B \times F \times F. \tag{54}$$

This implies that the normalized Poincaré bundle $\mathcal{P} \to X \times \widehat{X}$ splits as

$$\mathcal{P} \cong \mathcal{P}_B \boxtimes \mathcal{P}_F \tag{55}$$

where $\mathcal{P}_B$ and $\mathcal{P}_F$ are the normalized Poincaré bundles on $B \times B$ and $F \times F$ respectively. Since $\mathcal{P}$ is a line bundle with $c_1(\mathcal{P}) = c_1(\mathcal{P}_E) + c_1(\mathcal{P}_F)$, we can calculate the Chern classes of the Poincaré. We record the following expressions for the Chern classes:

$$\begin{aligned} c_1(\mathcal{P}) = {} & [\Delta_B \times F \times F] - [\mathit{0}_B \times B \times F \times F] - [B \times \mathit{0}_B \times F \times F] \\ & + [B \times B \times \Delta_F] - [B \times B \times \mathit{0}_F \times F] - [B \times B \times F \times \mathit{0}_F] \end{aligned} \tag{56}$$

$$\begin{aligned} c_1(\mathcal{P})^2 = 2\Big(& [\Delta_B \times \Delta_F] + [\mathit{0}_B \times B \times \mathit{0}_F \times F] + [\mathit{0}_B \times B \times F \times \mathit{0}_F] + [B \times \mathit{0}_B \times \mathit{0}_F \times F] \\ & + [B \times \mathit{0}_B \times F \times \mathit{0}_F] - [\mathit{0}_B \times \mathit{0}_B \times F \times F] - [B \times B \times \mathit{0}_F \times \mathit{0}_F] - [\Delta_B \times \mathit{0}_F \times F] \\ & - [\Delta_B \times F \times \mathit{0}_F] - [\mathit{0}_B \times B \times \Delta_F] - [B \times \mathit{0}_B \times \Delta_F]\Big) \end{aligned} \tag{57}$$

$$\begin{aligned} c_1(\mathcal{P})^3 = 6\Big(& [\mathit{0}_B \times \mathit{0}_B \times \mathit{0}_F \times F] + [\mathit{0}_B \times \mathit{0}_B \times F \times \mathit{0}_F] + [\mathit{0}_B \times B \times \mathit{0}_F \times \mathit{0}_F] \\ & + [B \times \mathit{0}_B \times \mathit{0}_F \times \mathit{0}_F] - [\mathit{0}_B \times \mathit{0}_B \times \Delta_F] - [\Delta_B \times \mathit{0}_F \times \mathit{0}_F]\Big). \end{aligned} \tag{58}$$

Let p and q denote the projection to $\widehat{X}$, and X respectively. With the Chern classes of $\mathcal{P}$, we can use Grothendieck-Riemann-Roch to calculate the following Chern character,

$$\mathrm{ch}\,\mathbf{R}\,\mathcal{S}(V) = p_*(\mathrm{ch}\,\mathcal{P}.\ q^*\,\mathrm{ch}\,V).$$

After multiplying the terms, we have that

$$c_1(\mathbf{R}\,\mathcal{S}(V)) = -d[\mathit{0}_B \times F] - \delta[B \times \mathit{0}_F] + \sum_i n_i\left([x_i \times F] - [\mathit{0}_B \times F] + [B \times y_i] - [B \times \mathit{0}_F]\right). \tag{59}$$

This class is fixed if and only if $\oplus_i n_i x_i = \oplus_i n_i 0_B = 0_B$, and $\oplus_i n_i y_i = \oplus_i n_i 0_F = 0_F$. Note that this is the addition on the elliptic curves B and F respectively. Hence fixing the determinant as $\det \mathbf{R}\mathcal{S}(V) = \mathcal{O}\left(-df - \delta\sigma\right)$, is equivalent to $\mathsf{a}(Z) = 0$. $\square$

Remark 1. *We note that this result was already known in* [MO5, Proposition 3.5]. *However the proof presented above is a novel approach using a calculation on the Chow group.*

2 New strange duality pairs

In this section we work on a product surface $X = B \times F$ which we view as a fibration over B. We set

$$\sigma = [B \times 0], \quad f = [0 \times F].$$

Our aim is to expand on Theorem 12 by using Fourier-Mukai kernels of higher rank. To define these kernels, we first fix a pair of positive coprime integers a and b. Let n and c be any integers that satisfy the equation

$$an + bc = 1. \tag{60}$$

By Lemma 1, there exists a unique vector bundle $\mathcal{U}_F \to F \times F$, with rank a and first Chern class,

$$c_1(\mathcal{U}_F) = c_1(\mathcal{P}_F) + b[0 \times F] + c[F \times 0]. \tag{61}$$

This vector bundle is semihomogeneous, so by Lemma 2 its Chern character is completely determined by its rank and first Chern class. It is given by the formula

$$\mathrm{ch}(\mathcal{U}_F) = a \exp\left(\frac{c_1(\mathcal{U}_F)}{a}\right).$$

Expanding this we get,

$$\mathrm{ch}\,\mathcal{U}_F = a + \Delta_F + (b-1)[0 \times F] + (c-1)[F \times 0] - n[0 \times 0]. \tag{62}$$

Note that when we restrict this bundle to the elliptic curve $F \times \{pt\}$ it has rank a and degree b. Similarly it restricts to a bundle of rank a and degree c on $\{pt\} \times F$. We consider the natural projection map

$\pi_F : B \times F_1 \times F_2 \to F_1 \times F_2$, and the bundle

$$\mathcal{U} = \pi_F^* (\mathcal{U}_F) \text{ on } B \times F_1 \times F_2.$$

Note that $F_1 \cong F_2 \cong F$; we label them for the sake of clarity. We let $X_1 = B \times F_1$ and $X_2 = B \times F_2$. We define the following Fourier-Mukai transform $\mathsf{FM}: D(X_1) \to D(X_2)$:

$$\mathsf{FM}(?) = \mathbf{R}\, p_* \left(\mathcal{U}^\vee \overset{\mathbf{L}}{\otimes} q^*(?) \right). \tag{63}$$

Here $p: B \times F_1 \times F_2 \to X_2$ is the projection, and likewise $q: B \times F_1 \times F_2 \to X_1$.

Proposition 8. *Fix a Mukai vector* $\widetilde{v} = (r, \sigma + \delta f, k) \in \mathrm{H}^{2*}(X_1)$ *with* $r \geq 2$.

(i) Given a locally free sheaf $\widetilde{V} \in \mathsf{M}_{\widetilde{v}}$, *its dual is WIT-1 with respect to* FM. *The map* $\widetilde{V} \mapsto V$, *where* $V[-1] = \mathsf{FM}(\widetilde{V}^\vee)$ *induces a birational map*

$$\mathsf{M}_{\widetilde{v}} \dashrightarrow \mathsf{M}_v,$$

where

$$v = \big(br + a, (rn - c)\sigma - (ak + \delta b) f, kc - \delta n\big) \in \mathrm{H}^{2*}(X_2). \tag{64}$$

This map is an isomorphism away from codimension 2.

(ii) Suppose that we fix the following Chern classes in Chow,

$$c_1(\widetilde{V}) = \sigma + \delta f \in \mathrm{A}^1(X_1), \quad c_2(\widetilde{V}) = Z = \sum_i n_i [x_i \times y_i] \in \mathrm{A}^2(X_1).$$

We let $Z_B = Z\,|_B$. *Then*

$$c_1(V) = (rn - c)\sigma - \delta(a + b) f + a[Z_B \times F] \in \mathrm{A}^1(X_2) \tag{65}$$

$$c_2(V) = \delta(c - n)[\mathit{0} \times \mathit{0}] + (1 - c)[Z_B \times \mathit{0}] - Z \in \mathrm{A}^2(X_2). \tag{66}$$

(iii) The birational map $\mathsf{M}_{\widetilde{v}} \dashrightarrow \mathsf{M}_v$ *restricts to a birational map*

$$\mathsf{K}_{\widetilde{v}} \dashrightarrow \mathsf{K}_v.$$

(Please note that here a sheaf $\widetilde{V}$ is in $\mathsf{K}_{\widetilde{v}}$ if and only if

$$\det \widetilde{V} = \mathcal{O}\left(\sigma + \delta f\right), \quad \det \mathbf{R}\,\mathcal{S}(\widetilde{V}) = \mathcal{O}\left(-\delta\sigma - f\right).)$$

Proof. Let $\widetilde{V} \in \mathsf{M}_{\widetilde{v}}$, be a locally free sheaf. Since $\mu(\widetilde{V}^\vee) = -1/r < b/a$, by Lemma 8 the sheaf $\widetilde{V}^\vee$ is WIT-1 with respect to FM. The dual, $\widetilde{V}^\vee$ is WIT-1 so we expect $\operatorname{rk} v < 0$. The fibers of p are isomorphic to the elliptic curve F, so we calculate that

$$\operatorname{rk} v = \chi\left(\mathcal{U}^\vee \otimes q^*(\widetilde{v}^\vee)\big|_F\right) = a(-1) + r(-b) = ar\left(-\frac{1}{r} - \frac{b}{a}\right).$$

The rank $\operatorname{rk} v$ is negative if

$$-1/r < b/a$$

which is always true since r, a and b are all positive integers. We have that

$$\mathsf{FM}(\widetilde{V}^\vee) = V[-1]$$

where V is a sheaf. This means that the map $\widetilde{V} \mapsto V$ is well defined.

As

$$\operatorname{ch} \widetilde{V} = r + (\sigma + \delta f) + \delta[0 \times 0] - Z \in A^*(X_1),$$

we calculate, via Grothendieck-Riemann-Roch,

$$\operatorname{ch} V = p_*\left(\operatorname{ch}(\mathcal{U}^\vee).q^*(\operatorname{ch} \widetilde{V}^\vee)\right) = p_*\left(\pi_F^* \operatorname{ch}(\mathcal{U}_F^\vee).q^*(\operatorname{ch} \widetilde{V}^\vee)\right).$$

Multiplying the terms in the above equation we find

$$\operatorname{ch}_0 V = -p_*\Big(- a[B \times 0 \times F] - r[B \times \Delta_F] - r(b-1)[B \times 0 \times F]\Big). \tag{67}$$

This simplifies to

$$\operatorname{ch}_0 V = br + a.$$

Similarly we have,

$$\begin{aligned}\operatorname{ch}_1 V = -p_*\Big(& a\delta[\mathit{0}\times \mathit{0}\times F] - a[Z\times F] - rn[B\times \mathit{0}\times \mathit{0}] \\ &+ [B\times \mathit{0}\times \mathit{0}] + (c-1)[B\times \mathit{0}\times \mathit{0}] + \delta[\mathit{0}\times \Delta_F] \\ &+ \delta(b-1)[\mathit{0}\times \mathit{0}\times F] + \delta(c-1)[\mathit{0}\times F\times \mathit{0}]\Big). \end{aligned} \tag{68}$$

This simplifies to

$$\operatorname{ch}_1 V = (rn-c)\sigma - \delta(a+b)f + a[Z_B\times F].$$

Finally we have,

$$\begin{aligned}\operatorname{ch}_2 V = -p_*\Big(& -\delta[\mathit{0}\times \mathit{0}\times \mathit{0}] + \sum_i n_i[x_i\times y_i\times y_i] \\ &-\delta(c-1)[\mathit{0}\times \mathit{0}\times \mathit{0}] + (c-1)[Z\times \mathit{0}] + \delta n[\mathit{0}\times \mathit{0}\times \mathit{0}].\Big) \end{aligned} \tag{69}$$

This last term simplifies to

$$\operatorname{ch}_2 V = \delta(c-n)[\mathit{0}\times \mathit{0}] + (1-c)[Z_B\times \mathit{0}] - Z.$$

Note that we multiply all the pushforwards by -1 since the sheaf is WIT-1. This completes the proof of *(ii)*. Under the cycle class map, we have that

$$[Z_B\times F] = \ell(Z)f, \quad [Z_B\times \mathit{0}] = \ell(Z)[\mathit{0}\times \mathit{0}].$$

Given that

$$k = \delta - \ell(Z),$$

the class in Chow maps to the Mukai vector

$$v = \big(br+a, (rn-c)\sigma - (ak+\delta b)f, kc - \delta n\big)$$

in cohomology. The fact that FM induces a birational map follows from the explicit construction of the inverse map

$$\mathsf{FM}^{-1}(?) = \mathbf{R}\, q_* \left(\mathcal{U}[1] \overset{\mathbf{L}}{\otimes} p^*(?) \right). \tag{70}$$

In the proof of [BH, Theorem 5.1] the authors show the birational map on moduli spaces induced by the Fourier-Mukai transform FM is an isomorphism away from codimension 2, assuming that $r \geq 3$. On the other hand in [BMOY, Remark 3.1] shows that the Fourier-Mukai transform is regular in codimension 1 when the rank $r \geq 2$. This completes the proof of *(i)*. Suppose the $\widetilde{V} \in \mathsf{K}_{\widetilde{v}}$ is a locally free sheaf with Chern character η. Then by Proposition 7 we have that $\mathsf{a}(Z) = 0$, but this implies that $\mathsf{a}(\mathrm{ch}_2 V) = 0$ and so again by the lemma we obtain $V \in \mathsf{K}_v$. □

We note that in [BH, §3] the authors mention that the theorem assumes that X is a K3 surface, but that the results hold for any elliptic surface that has at most finitely many nodal fibers. This is the case for an elliptic abelian surface. Please also see [BM] for more details in the K3 setting, as well as [YY] for more details in the abelian setting.

Proposition 9. *Fix a Mukai vector $\widetilde{v} = (r, \sigma + \delta f, k) \in \mathrm{H}^{2*}(X_1)$, with $r \geq 2$, such that*

$$b/a < 1/r. \tag{71}$$

(i) Given a locally free sheaf $\widetilde{V} \in \mathsf{M}_{\widetilde{v}}$, then $V = \mathsf{FM}(\widetilde{V})^\vee$ is a sheaf. The map $\widetilde{V} \mapsto V$ induces a birational map

$$\mathsf{M}_{\widetilde{v}} \dashrightarrow \mathsf{M}_v,$$

where

$$v = \big(a - rb, (rn + c)\sigma + (\delta b - ka)f, -(kc + \delta n)\big) \in \mathrm{H}^{2*}(X_2). \tag{72}$$

This map is an isomorphism away from codimension 2.

(ii) Suppose that we fix the following Chern classes in Chow,

$$c_1(\widetilde{V}) = \sigma + \delta f \in \mathrm{A}^1(X_1), \quad c_2(\widetilde{V}) = Z = \sum_i n_i [x_i \times y_i] \in \mathrm{A}^2(X_1).$$

We let $Z_B = Z|_B$, *then*

$$c_1(V) = (rn + c)\sigma + \delta(b-a)f + a[Z_B \times F] \in \mathrm{A}^1(X_2) \tag{73}$$

$$c_2(V) = -\delta(n+c)[0 \times 0] + (c-1)[Z_B \times 0] + Z \in \mathrm{A}^2(X_2). \tag{74}$$

(iii) The birational map $\mathsf{M}_{\widetilde{v}} \dashrightarrow \mathsf{M}_v$ *restricts to a birational map*

$$\mathsf{K}_{\widetilde{v}} \dashrightarrow \mathsf{K}_v.$$

(Note as before that a sheaf $\widetilde{V}$ *is in* $\mathsf{K}_{\widetilde{v}}$ *if and only if*

$$\det \widetilde{V} = \mathcal{O}\left(\sigma + \delta f\right), \quad \det \mathbf{R}\,\mathcal{S}(\widetilde{V}) = \mathcal{O}\left(-\delta\sigma - f\right).)$$

Proof. By definition the derived dual

$$\mathsf{FM}(\widetilde{V})^\vee = \mathbf{R}\,\mathcal{H}\mathrm{om}(\mathsf{FM}(\widetilde{V}), \mathcal{O}_X) = \mathbf{R}\,\mathcal{H}\mathrm{om}\left(\mathbf{R}\,p_*(\mathcal{U}^\vee \otimes q^*\widetilde{V}), \mathcal{O}_X\right).$$

By Grothendieck duality this is isomorphic to

$$\mathbf{R}\,p_*\,\mathbf{R}\,\mathcal{H}\mathrm{om}\left(\mathcal{U}^\vee \otimes q^*\widetilde{V}, \mathcal{O}_{F\times X}[1]\right) \cong \mathbf{R}\,p_*(\mathcal{U} \otimes q^*\widetilde{V}^\vee)[1].$$

We will show that $\mathbf{R}\,p_*(\mathcal{U} \otimes q^*\widetilde{V}^\vee) = V[-1]$, where V is a sheaf. This will imply that $\mathsf{FM}(\widetilde{V})^\vee = V$ is a sheaf. By assumption $\mu(\widetilde{V}^\vee) = -1/r < -b/a$, and so by Lemma 8 we conclude that $\mathbf{R}\,p_*(\mathcal{U} \otimes q^*\widetilde{V}^\vee) = V[-1]$. We also calculate the rank

$$\mathrm{rk}\,v = \chi\left(\mathcal{U} \otimes q^*\widetilde{V}^\vee|_F\right) = a(-1) + rb = ar\left(-\frac{1}{r} + \frac{b}{a}\right).$$

We find that $\mathrm{rk}\,v < 0$, as expected. Now we use Grothendieck-Riemann-Roch to calculate

$$\mathrm{ch}\,V = p_*(\mathrm{ch}(\mathcal{U}) \cdot q^*\mathrm{ch}\,\widetilde{V}^\vee).$$

We already calculated that

$$\operatorname{ch}_0 V = a - rb. \tag{75}$$

We find

$$\operatorname{ch}_1 V = -p_*\Big(- rn[B \times \mathit{0} \times \mathit{0}] + a\delta[\mathit{0} \times \mathit{0} \times F] - a[Z \times F] - (c-1)\delta\,[\mathit{0} \times F \times \mathit{0}] \\ - [B \times \mathit{0} \times \mathit{0}] - (c-1)[B \times \mathit{0} \times \mathit{0}] - \delta[\mathit{0} \times \Delta_F] - \delta(b-1)[\mathit{0} \times \mathit{0} \times F]\Big),$$

which yields

$$\operatorname{ch}_1 V = (rn + c)\sigma + \delta(b-a)f + a[Z_B \times F]. \tag{76}$$

Similarly we find

$$\operatorname{ch}_2 V = -p_*\Big(n\delta[\mathit{0} \times \mathit{0} \times \mathit{0}] + \delta[\mathit{0} \times \mathit{0} \times \mathit{0}] + \delta(c-1)[\mathit{0} \times \mathit{0} \times \mathit{0}] \\ - \sum_i n_i[x_i \times y_i \times y_i] - (c-1)[Z \times \mathit{0}]\Big),$$

which simplifies to

$$\operatorname{ch}_2 V = -\delta(n+c)[\mathit{0} \times \mathit{0}] + (c-1)[Z_B \times \mathit{0}] + Z. \tag{77}$$

This completes the proof of *(ii)*. As in the proof of the previous proposition, under the cycle class map we have,

$$[Z_B \times F] = \ell(Z)f, \quad [Z_B \times \mathit{0}] = \ell(Z)[\mathit{0} \times \mathit{0}].$$

Letting $k = \delta - \ell(Z)$, we find that

$$v = \big(a - rb, (rn + c)\sigma + (\delta b - ka)f, -(kc + \delta n)\big).$$

The fact that the induced map $\mathsf{M}_{\widetilde{v}} \dashrightarrow \mathsf{M}_v$ on moduli spaces is birational and an isomorphism away from codimension 2 follows exactly as in the proof of the previous proposition. This completes the proof of *(i)*. Finally to prove *(iii)*, by Proposition 7 if $\widetilde{V} \in \mathsf{K}_{\widetilde{v}}$, we know that $\mathsf{a}(Z) = \mathit{0}$, but this implies that $\mathsf{a}(\operatorname{ch}_2 V) = \mathit{0}$, and so $V \in \mathsf{K}_v$. ∎

We are now in a position to prove the following theorem, which follows directly from the previous two propositions. This theorem will afterwards be reformulated as Theorem 1 of the thesis.

Theorem 13. *Let $X = B \times F$ be a product of elliptic curves. Choose a pair of positive coprime integers a and b satisfying*

$$an + bc = 1. \tag{78}$$

Fix the data of two orthogonal Mukai vectors of the form

$$\widetilde{v} = (r, \sigma + \delta f, k), \quad \widetilde{w} = (r', \sigma + \delta' f, k')$$

with ranks $r, r' \geq 2$. We require that

$$1/r' > b/a. \tag{79}$$

Given these data, consider the following Mukai vectors

$$v = \big(br + a, (rn - c)\sigma - (ak + \delta b)f, kc - \delta n\big) \tag{80}$$

$$w = \big(a - r'b, (r'n + c)\sigma + (\delta' b - k'a)f, -(k'c + \delta' n)\big). \tag{81}$$

Then the locus

$$\Theta = \left\{(V, W) : \mathrm{H}^1(V \overset{\boldsymbol{L}}{\otimes} W) \neq 0\right\} \subset \mathsf{K}_v \times \mathsf{M}_w$$

is a Cartier divisor inducing an isomorphism

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v).$$

Proof. The choice of integers $\{a, b, c, n\}$ allows us to construct the bundle $\mathcal{U}$ with Chern character

$$\mathrm{ch}\,\mathcal{U} = a + \Delta_F + (b-1)[0 \times F] + (c-1)[F \times 0] - n[0 \times 0].$$

With this vector bundle $\mathcal{U}$, we define the following Fourier-Mukai transform

$$\mathsf{FM}\colon D(X) \to D(X), \quad \mathsf{FM}(?) = \mathbf{R}\, p_*\left(\mathcal{U}^\vee \overset{\mathbf{L}}{\otimes} q^*(?)\right).$$

Given a locally free sheaf $\widetilde{V} \in \mathsf{K}_{\widetilde{v}}$, by Proposition 8 we know that $\mathsf{FM}(\widetilde{V}^\vee) = V[-1]$ for a sheaf V with Mukai vector v as in (80). Similarly, since we assume that $1/r' > b/a$, by Proposition 9 we know that given a locally free sheaf $\widetilde{W} \in \mathsf{M}_{\widetilde{w}}$, that $\mathsf{FM}(\widetilde{W}) = W^\vee$ for a sheaf W with Mukai vector w as in (81). By Theorem 12 the locus

$$\widetilde{\Theta} = \left\{ (\widetilde{V}, \widetilde{W}) : \mathrm{H}^0(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}) \neq 0 \right\} \subset \mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}}$$

is a divisor inducing an isomorphism

$$\mathsf{SD}\colon \mathrm{H}^0(\mathsf{K}_{\widetilde{v}}, \Theta_{\widetilde{w}})^\vee \to \mathrm{H}^0(\mathsf{M}_{\widetilde{w}}, \Theta_{\widetilde{v}}).$$

Consider the following calculation,

$$\begin{aligned}
\mathrm{H}^0(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}) &\cong \mathrm{Hom}_{D(X)}\left(\widetilde{V}^\vee, \widetilde{W}\right) \\
&\cong \mathrm{Hom}_{D(X)}\left(\mathsf{FM}(\widetilde{V}^\vee), \mathsf{FM}(\widetilde{W})\right) \cong \mathrm{Hom}_{D(X)}\left(V[-1], W^\vee\right) \\
&\cong \mathrm{Ext}^1(V, W^\vee) \cong \mathrm{H}^1(V \otimes W)^\vee.
\end{aligned}$$

This calculation shows that

$$\mathrm{H}^0(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}) \neq 0 \iff \mathrm{H}^1(V \overset{\mathbf{L}}{\otimes} W) \neq 0.$$

By the previous two propositions, the Fourier-Mukai transform induces a birational map

$$\Psi\colon \mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}} \dashrightarrow \mathsf{K}_v \times \mathsf{M}_w$$

which is an isomorphism away from codimension 2. Since the map is regular in codimension 1 we can identify the divisor $\widetilde{\Theta}$ with the divisor

$$\Theta = \left\{ (V, W) : \mathrm{H}^1(V \overset{\mathbf{L}}{\otimes} W) \neq 0 \right\} \subset \mathsf{K}_v \times \mathsf{M}_w.$$

Since $\widetilde{\Theta}$ induces the strange duality isomorphism, it follows that Θ also induces the isomorphism

$$\mathsf{SD}\colon \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v).$$

□

3 Proof of Theorem 1

The ranks and the fiber degrees of the new pairs of Mukai vectors for which strange duality was proven to hold in Theorem 13, satisfy simple numerical relations. These allow us to reformulate Theorem 13 as Theorem 1 of the thesis.

Proof of Theorem 1. We begin by assuming that $b > 0$, and fix a pair of coprime integers (n, b). Let (α, β) be a pair of integers with $\alpha > 0$ such that

$$\alpha n + \beta b = 1. \tag{82}$$

The pair (α, β) represents a general choice for the rank and fiber degree of the vector w. For any $r' \geq 2$ we can set

$$a = \alpha + r'b, \quad c = \beta - r'n.$$

Note that again we have that

$$an + bc = 1.$$

We may assume without loss of generality that $r' = 2$, which allows us to solve for the pair (a, c). The choice of integers $\{a, b, c, n\}$ allows us to construct the bundle $\mathcal{U}$ with Chern character

$$\mathrm{ch}\,\mathcal{U} = a + \Delta_F + (b-1)[0 \times F] + (c-1)[F \times 0] - n[0 \times 0].$$

With this vector bundle $\mathcal{U}$, we define the Fourier-Mukai transform

$$\mathsf{FM}\colon D(X) \to D(X), \quad \mathsf{FM}(?) = \mathbf{R}\, p_* \left(\mathcal{U}^\vee \overset{\mathbf{L}}{\otimes} q^*(?) \right).$$

If we let $\widetilde{w}$ be any vector with $\mathrm{rk}\, w \geq 2$, and $\mathrm{fdeg}\, w = 1$, as in the previous Theorem 13, we can achieve any

$$(\alpha, \beta) = (\mathrm{rk}\, w, \mathrm{fdeg}\, w)$$

which satisfy (5) by using FM as in Theorem 13. After fixing these choices, note that a general solution to (4) is given by

$$\mathrm{rk}\, v = a + br, \quad \mathrm{fdeg}\, v = -c + rn.$$

The inequality (6) is satisfied if and only if $r + r' \geq 4$ which holds if and only if $r \geq 2$. Again this corresponds to a Mukai vector $\widetilde{v}$ as in Theorem 13.

Now suppose that $b < 0$, then we immediately have by sign reasons that $1/r' > b/a$. Assuming that $\operatorname{rk} v > 0$ gives us the inequality

$$-1/r < b/a$$

which is necessary in the assumptions of Theorem 13. We carry out the same analysis as in the case when $b > 0$. The only difference now being we start by fixing our choices for the Mukai vector v. Following the same argument as when $b > 0$ we arrive at the same conclusion, namely that we can produce any pair of Mukai vectors satisfying Equations (4), (5) and the constraint (6). $\square$

4 Alternate derivation of Theorem 1

In the previous section we derived Theorem 1 by starting with an orthogonal pair of Mukai vectors that satisfy the hypotheses of Theorem 12 for split surfaces. Then we transformed the pair of vectors to a new pair using a kernel that had first Chern class

$$c_1(\mathcal{U}_F) = c_1(\mathcal{P}_F) + b[0 \times F] + c[F \times 0].$$

In this section we take a slightly different approach, we begin with a general pair of vectors and try to apply conditions to them so that they transform into a pair of vectors that satisfy the hypotheses of Theorem 12.

The following two propositions are direct parallels to Propositions 8 and 9, now allowing arbitrary fiber degrees. They capture the behavior of a locally free sheaf with respect to the Fourier-Mukai transform defined in (63) with kernel given by (62). In particular the transform will send a locally free sheaf V to either a sheaf shifted by $[-1]$ or the dual of a sheaf. This depends on if the slope $\mu(V)$ is less than b/a or greater than b/a respectively.

Proposition 10. *Fix a Mukai vector $v = (r, d\sigma + \delta f, \chi) \in \mathrm{H}^{2*}(X_1)$, and let $V \in \mathsf{M}_v$ be a locally free sheaf such that $\mu(V) = d/r < b/a$.*

(i) Then V is WIT-1 with respect to FM, and the map $V \mapsto \widetilde{V}$, where $\widetilde{V}[-1] = \mathsf{FM}(V)$, induces a birational morphism

$$\Psi^{<} \colon \mathsf{M}_v \dashrightarrow \mathsf{M}_{\widetilde{v}}, \tag{83}$$

where

$$\widetilde{v} = (br - ad, (rn + dc)\sigma + (\delta b - \chi a)f, n\delta + \chi c) \in \mathrm{H}^{2*}(X_2). \tag{84}$$

(ii) The birational map $\Psi^{<} \colon \mathsf{M}_v \dashrightarrow \mathsf{M}_{\widetilde{v}}$ *restricts to a birational map which we also denote*

$$\Psi^{<} \colon \mathsf{K}_v \dashrightarrow \mathsf{K}_{\widetilde{v}}.$$

(Please note that here a sheaf V *is in* K_v *if and only if*

$$\det V = \mathcal{O}\left(d\sigma + \delta f\right), \quad \det \mathbf{R}\,\mathcal{S}(V) = \mathcal{O}\left(-\delta\sigma - df\right)).$$

Proof. Let $V \in \mathsf{M}_v$ be a locally free sheaf. Since $\mu(V) = d/r < b/a$, by Lemma 8 the sheaf V is WIT-1 with respect to FM. The fibers of p are isomorphic to the elliptic curve F, so we calculate that

$$\operatorname{rk}\mathsf{FM}(V) = \chi\left(\mathcal{U}^{\vee} \otimes q^{*}V\big|_{F}\right) = ad - rb = ar\left(\frac{d}{r} - \frac{b}{a}\right).$$

Since a and r are both ranks, and hence positive, the above rank is negative as we expect since the sheaf is WIT-1. We use Grothendieck-Riemann-Roch to calculate

$$\operatorname{ch}\mathsf{FM}(V) = p_{*}\left(\operatorname{ch}(\mathcal{U}^{\vee}).q^{*}\operatorname{ch}(V)\right) = p_{*}\left(\pi_F^{*}\operatorname{ch}(\mathcal{U}_F^{\vee}).q^{*}\operatorname{ch}(V)\right).$$

The calculation proceeds exactly as in the case of Proposition 8. $\square$

Proposition 11. *Fix a Mukai vector* $v = (r, d\sigma + \delta f, \chi) \in \mathrm{H}^{2*}(X_1)$*, and let* $V \in \mathsf{M}_v$ *be a locally free sheaf such that* $\mu(V) = d/r > b/a$.

(i) Then $\mathsf{FM}(V)^{\vee} = \widetilde{V}$ *is a sheaf. The map* $V \mapsto \widetilde{V}$ *induces a birational morphism*

$$\Psi^{>} \colon \mathsf{M}_v \dashrightarrow \mathsf{M}_{\widetilde{v}}, \tag{85}$$

where

$$\widetilde{v} = (ad - br, (nr + dc)\sigma + (\delta b - a\chi)f, -(n\delta + \chi c)) \in \mathrm{H}^{2*}(X_2). \tag{86}$$

(ii) The birational map $\Psi^{>} : \mathsf{M}_v \dashrightarrow \mathsf{M}_{\widetilde{v}}$ *restricts to a birational map which we also denote*

$$\Psi^{>} : \mathsf{K}_v \dashrightarrow \mathsf{K}_{\widetilde{v}}.$$

(Please note that here a sheaf V *is in* K_v *if and only if*

$$\det V = \mathcal{O}\left(d\sigma + \delta f\right), \quad \det \mathbf{R}\,\mathcal{S}(V) = \mathcal{O}\left(-\delta\sigma - df\right)).$$

Proof. The proof is completely the same as in Proposition 9, except with a coefficient of d instead of 1 in front of σ. □

Alternate Proof of Theorem 1. We begin by letting

$$r = \operatorname{rk} v, \quad d = \operatorname{fdeg} v.$$

Note that a general solution (R, D) to the equation

$$nR + cD = 1 \tag{87}$$

is given by,

$$R = r + c\lambda \tag{88}$$

$$D = -d - n\lambda \tag{89}$$

for any integer λ. This is because by assumption $(r, -d)$ is a particular solution to the linear Diophantine equation (87). If we let $r' = \operatorname{rk} w$ and $d' = \operatorname{fdeg} w$, then

$$r' = r + ck, \quad d' = -d - nk$$

for some nonzero integer k. Recall that we denote the class of a section and fiber of $X \to B$ respectively as

$$\sigma = [B \times \mathit{0}], \quad f = [\mathit{0} \times F].$$

With this in mind we write the Mukai vectors v and w explicitly as

$$v = (r, d\sigma + \delta f, \chi) \tag{90}$$

$$w = (r', d'\sigma + \delta' f, \chi'). \tag{91}$$

Suppose first that $c > 0$ and that $k < 0$. Then by (6) we have that

$$k = \frac{r' - r}{c} \leq -4.$$

In this case, we pick an integer $\ell \leq -2$ such that

$$\ell - k \geq 2 \tag{92}$$

and we define the following two integers

$$a = r + \ell c, \tag{93}$$

$$b = -d - \ell n. \tag{94}$$

Note that these satisfy the equation (87), and so $an + bc = 1$. Then by Lemma 1, there exists a unique vector bundle $\mathcal{U}_F \to F \times F$ with rank a and first Chern class,

$$c_1(\mathcal{U}_F) = c_1(\mathcal{P}_F) + b[\mathit{0} \times F] + c[F \times \mathit{0}]. \tag{95}$$

Since this bundle is semihomogeneous its Chern character is determined by its rank and first Chern class, and is given by

$$\mathrm{ch}\,\mathcal{U}_F = a + \Delta_F + (b-1)[\mathit{0} \times F] + (c-1)[F \times \mathit{0}] - n[\mathit{0} \times \mathit{0}]. \tag{96}$$

We define $\mathcal{U} = \pi_F^*(\mathcal{U}_F)$, where $\pi_F \colon B \times F \times F \to F \times F$ is the natural projection. We let

$$\mathsf{FM}(?) = \mathbf{R}\,p_*\left(\mathcal{U}^\vee \overset{\mathbf{L}}{\otimes} q^*(?)\right),$$

as was defined in (63). By our choice of integer ℓ, note that if $V \in \mathsf{K}_v$ is a locally free sheaf then

$$\mu(V^\vee) = -d/r < b/a \iff (rn - cd)\ell < 0 \iff \ell < 0.$$

The last implication coming from the fact that $rn - cd = 1$. Similarly note that if $W \in \mathsf{M}_w$ is a locally free sheaf then

$$\mu(W) = d'/r' > b/a \iff (rn - cd)(\ell - k) > 0 \iff \ell > k.$$

By our choice of ℓ as in (92) both of these inequalities are satisfied. Then by Proposition 10, $\mathsf{FM}\,(V^\vee) = \widetilde{V}[-1]$ and we have a birational morphism

$$\Psi^<\colon \mathsf{K}_v \to \mathsf{K}_{\widetilde{v}}$$

where

$$\widetilde{v} = (br + ad, (rn - dc)\sigma + (-\delta b - \chi a)f, -n\delta + \chi c).$$

Notice that

$$\operatorname{rk}\widetilde{v} = br - ad = -\ell(rn - cd) = -\ell \geq 2,$$

and that

$$\operatorname{fdeg}\widetilde{v} = rn - dc = 1.$$

Similarly by Proposition 11, $\mathsf{FM}(W) = \widetilde{W}^\vee$ and we have a birational morphism

$$\Psi^>\colon \mathsf{M}_w \dashrightarrow \mathsf{M}_{\widetilde{w}}$$

where

$$\widetilde{w} = (ad' - br', (nr' + d'c)\sigma + (\delta' b - a\chi')f, -(n\delta' + \chi' c)).$$

Likewise we have that

$$\operatorname{rk}\tilde{w} = \ell - k \geq 2$$

where the last inequality comes from our definition of ℓ in (92). Similarly

$$\operatorname{fdeg}\widetilde{w} = r'n + d'c = 1$$

since they are solutions to (87). Consider the following calculation,

$$\begin{aligned}
\mathrm{H}^1\left(V \overset{\mathbf{L}}{\otimes} W\right) &\cong \mathrm{Hom}_{D(X)}\left(V^\vee, W[1]\right) \\
&\cong \mathrm{Hom}_{D(X)}\left(\mathsf{FM}(V^\vee), \mathsf{FM}(W)[1]\right) \cong \mathrm{Hom}_{D(X)}\left(\widetilde{V}[-1], \widetilde{W}^\vee[1]\right) \\
&\cong \mathrm{Hom}_{D(X)}\left(\widetilde{V}, \widetilde{W}^\vee[2]\right) \overset{\text{Serre}}{\cong} \mathrm{Hom}_{D(X)}\left(\widetilde{W}^\vee, \widetilde{V}\right)^* \\
&\cong \mathrm{H}^0\left(\widetilde{W} \otimes \widetilde{V}\right)^*
\end{aligned}$$

From this calculation we conclude that

$$\mathrm{H}^1\left(V \overset{\mathbf{L}}{\otimes} W\right) \neq 0 \iff \mathrm{H}^0\left(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}\right) \neq 0.$$

Note that the labeled isomorphism comes from Serre duality 3, since ω_X is trivial for an abelian variety. Note that the Mukai vectors $\widetilde{v}$ and $\widetilde{w}$ satisfy the hypotheses of Theorem 12, so the divisor

$$\widetilde{\Theta} = \left\{(\widetilde{V}, \widetilde{W}) : \mathrm{H}^0\left(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}\right) \neq 0\right\} \subset \mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}}$$

induces the strange duality morphism. By [BMOY, Remark 3.1] since the ranks of the Mukai vectors $\widetilde{v}$ and $\widetilde{w}$ are at least 2 the inverse transformation FM^{-1} is an isomorphism away from codimension 2, and so in particular FM is as well. This means that we can identify the divisor $\widetilde{\Theta}$ with the divisor Θ in (7). Since $\widetilde{\Theta}$ induces the strange duality morphism, then so too does Θ.

Now suppose that $c > 0$ and that $k > 0$, then by (6) we have that $k \geq 4$. In this case we pick an integer $\ell \geq 2$ such that

$$k - \ell \geq 2. \tag{97}$$

We use this choice of ℓ to and define a pair of integers a and b as in Equations (93) and (94) respectively. This defines for us a kernel, with respect to which we take our transform FM. In this case

$$\mu(V^\vee) > b/a \iff \ell > 0$$

which is true by our choice of ℓ. Analogously we have that

$$\mu(W) < b/a \iff \ell < k$$

which also holds. So by Propositions 11 and 10 respectively we have a birational morphism

$$\Psi^{>} \times \Psi^{<} \colon \mathsf{K}_v \times \mathsf{M}_w \dashrightarrow \mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}}$$

where

$$\widetilde{v} = (-ad - br, (nr - dc)\sigma + (-\delta b - a\chi)f, -(-n\delta + \chi c)),$$

$$\widetilde{w} = (br' - ad', (r'n + d'c)\sigma + (\delta' b - \chi' a)f, n\delta' + \chi' c).$$

Notice again that the fiber degrees of both these new Mukai vectors are given by solutions to (87) and so they are both Mukai vectors with fiber degree 1. Finally note that $\operatorname{rk}\widetilde{v} = \ell \geq 2$, and $\operatorname{rk}\widetilde{w} = k - \ell \geq 2$, and so again they satisfy the hypotheses of Theorem 12 so the divisor $\widetilde{\Theta}$ induces strange duality. We consider the following calculation,

$$\begin{aligned}
\mathrm{H}^1\left(V \overset{\mathbf{L}}{\otimes} W\right) &\cong \operatorname{Hom}_{D(X)}\left(V^\vee, W[1]\right) \\
&\cong \operatorname{Hom}_{D(X)}\left(\mathsf{FM}(V^\vee), \mathsf{FM}(W)[1]\right) \cong \operatorname{Hom}_{D(X)}\left(\widetilde{V}^\vee, \widetilde{W}^\vee[1-1]\right) \\
&\cong \operatorname{Hom}_{D(X)}\left(\widetilde{V}^\vee, \widetilde{W}\right) \cong \mathrm{H}^0\left(\widetilde{V} \otimes \widetilde{W}\right)
\end{aligned}$$

This calculation shows that

$$\mathrm{H}^1\left(V \overset{\mathbf{L}}{\otimes} W\right) \neq 0 \iff \mathrm{H}^0\left(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}\right) \neq 0.$$

As before this means that we can identify the divisor $\widetilde{\Theta}$ with the divisor Θ in (7). Since $\widetilde{\Theta}$ induces the strange duality morphism, then so too does Θ.

Suppose now that $c < 0$ and that $k > 0$, then we choose our ℓ as in the second case we considered, given by (97). This implies that $\mu(V^\vee) > b/a$, and that $\mu(W) < b/a$. So with the exact same reasoning as before the divisor Θ induces the strange duality morphism. If we are in the situation where $c < 0$ and $k < 0$ then we choose our ℓ as in the first case given by (92). Just as before this implies that $\mu(V^\vee) < b/a$, and that

$\mu(W) > b/a$. With the exact same analysis as was carried out in the first case, we conclude that the divisor Θ induces the strange duality morphism.

$\square$

5 Proof of Theorem 2

Theorem 2 is a specialization of Theorem 1, where we make an explicit choice for our kernels. This theorem gives a slightly different procedure on how to produce pairs of Mukai vectors that exhibit the strange duality isomorphism.

Proof of Theorem 2. We use the data of (x, y, α, β) to construct the vector bundle $\mathcal{U}$ with Chern character

$$\operatorname{ch}(\mathcal{U}) = (yr' + x) + \Big(c_1(\mathcal{P}_F) + y[0 \times F] + \big(\beta - xt - r'(\alpha + yt)\big)[F \times 0]\Big) \\ - (\alpha + yt)[0 \times 0] \in A^*(X).$$

We let $\mathcal{U}^\vee$ be the kernel used in the Fourier-Mukai transform FM. We also have the data of a pair of orthogonal Mukai vectors $\widetilde{v}$ and $\widetilde{w}$, both with rank at least 2. By Theorem 12 the locus

$$\widetilde{\Theta} = \left\{(\widetilde{V}, \widetilde{W}) : \mathrm{H}^0(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}) \neq 0\right\} \subset \mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}}$$

is a divisor inducing an isomorphism

$$\mathsf{SD}\colon\ \mathrm{H}^0(\mathsf{K}_{\widetilde{v}}, \Theta_{\widetilde{w}})^\vee \to \mathrm{H}^0(\mathsf{M}_{\widetilde{w}}, \Theta_{\widetilde{v}}).$$

Given a locally free sheaf $\widetilde{V} \in \mathsf{K}_{\widetilde{v}}$, by Proposition 8 we know that $\mathsf{FM}(\widetilde{V}^\vee) = V[-1]$ for a sheaf $V \in \mathsf{K}_v$ with v as in (9). If we let $a = yr' + x$ and $b = y$, we have by construction that the following inequality is satisfied

$$b/a < 1/r'.$$

So given a locally free sheaf $\widetilde{W} \in \mathsf{M}_{\widetilde{w}}$, by Proposition 9 we know that $\mathsf{FM}(\widetilde{W}) = W^\vee$ for a sheaf W with Mukai vector w as in (10). By the previous two propositions, the Fourier-Mukai transform induces a birational morphism

$$\mathsf{K}_{\widetilde{v}} \times \mathsf{M}_{\widetilde{w}} \dashrightarrow \mathsf{K}_v \times \mathsf{M}_w$$

that is an isomorphism away from codimension 2. As a result, we can identify the divisor $\widetilde{\Theta}$ with the divisor Θ in the following way,

$$\begin{aligned} 0 \neq \mathrm{H}^0(\widetilde{V} \overset{\mathbf{L}}{\otimes} \widetilde{W}) &\cong \mathrm{Hom}_{D(X)}\left(\widetilde{V}^\vee, \widetilde{W}\right) \\ &\cong \mathrm{Hom}_{D(X)}\left(\mathsf{FM}(\widetilde{V}^\vee), \mathsf{FM}(\widetilde{W})\right) \cong \mathrm{Hom}_{D(X)}\left(V[-1], W^\vee\right) \\ &\cong \mathrm{Ext}^1(V, W^\vee) \cong \mathrm{H}^1(V \otimes W)^\vee. \end{aligned}$$

So if we let

$$\Theta = \left\{(V, W) : \mathrm{H}^1\left(V \overset{\mathbf{L}}{\otimes} W\right) \neq 0\right\} \subset \mathsf{K}_v \times \mathsf{M}_w$$

then this divisor induces an isomorphism

$$\mathsf{SD}\colon \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v).$$

□

6 Corollaries of the main theorems

In this section we apply the two main theorems of the thesis and arrive at a number of interesting classes of Mukai vector pairs where strange duality is shown to hold. We start with the proof of Corollary 1 given in the Introduction. We follow this with some more corollaries.

Proof of Corollary 1. We let $R = 2r + 1$ for some integer $r \geq 2$, and consider the following pair of orthogonal vectors $\widetilde{v} = (r, \sigma - (kr)f, k)$ and $\widetilde{w} = (r, \sigma - (k'r)f, k')$. These Mukai vectors satisfy the hypotheses of the input vectors in Theorem 2. We also let

$$(x, y, \alpha, \beta) = (1, 1, 1, 0), \quad t = s - 1.$$

And so by Theorem 2 we arrive at the above pair of Mukai vectors with strange duality holding between the corresponding moduli spaces of sheaves. □

We now state another corollary which gives strange duality for a pair of Mukai vectors with fiber degrees 1 and -1.

Corollary 3. *Let*

$$r \geq 2, \quad k, k' < 0$$

and consider the following pair of Mukai vectors,

$$v = \Big(2r+1, -\sigma - kf, k\Big) \tag{98}$$

$$w = \Big(1, \sigma - k'(2r+1)f, -k'\Big). \tag{99}$$

The map

$$\mathsf{SD}\colon \mathrm{H}^0(\mathsf{K}_v, \Theta_w)^\vee \to \mathrm{H}^0(\mathsf{M}_w, \Theta_v)$$

is an isomorphism.

Proof. Consider the following pair of orthogonal vectors $\widetilde{v} = (r, \sigma - (kr)f, k)$ and $\widetilde{w} = (r, \sigma - (k'r)f, k')$. These Mukai vectors satisfy the hypotheses of the input vectors in Theorem 2. We also let

$$(x, y, \alpha, \beta) = (1, 1, 1, 0), \quad t = -1.$$

And so by Theorem 2 we arrive at the above pair of Mukai vectors with strange duality holding between the corresponding moduli spaces of sheaves. □

V. Fourier-Mukai transforms on nonsplit elliptic abelian surfaces

1 Elliptic abelian surfaces and their Fourier-Mukai partners

In this section, we consider a primitively polarized abelian surface (X, H) with $H^2 = 2d$ containing an elliptic curve $F \hookrightarrow X$ as a subgroup. We set

$$d_F = H \cdot F.$$

We further let

$$B = X/F$$

and view X as an elliptic abelian surface over the base B,

$$\pi : X \to B.$$

By the Poincaré Reducibility Theorem (Theorem 6 in Chapter 2), there exists a uniquely specified complementary elliptic curve $E \hookrightarrow X$, and the addition map

$$\tau_X : E \times F \to X \tag{100}$$

is an isogeny. We assume throughout this chapter that E and F are not isogenous. We set

$$d_E = H \cdot E.$$

By (22) in Theorem 6, the degree δ of the isogeny τ_X is

$$\delta = \#E \cap F = E \cdot F = \frac{d_E \cdot d_F}{d}. \tag{101}$$

As we now have the diagram

$$\begin{tikzcd} F \arrow[r, hook] & X \arrow[d, "\pi"'] & E \arrow[l, hook'] \\ & B. & \end{tikzcd}$$

we let $\tau\colon E \to B$ denote the composition of the inclusion $E \hookrightarrow X$ and the projection $X \to B$.

Lemma 9. *The following diagram is a fiber square:*

$$\begin{tikzcd} E \times F \arrow[r] \arrow[d, "\tau_X"'] & E \arrow[d, "\tau"] \\ X \arrow[r, "\pi"] & B. \end{tikzcd}$$

Here the top horizontal arrow is the projection onto the first factor.

Proof. We consider the map

$$E \times F \to X \times_B E, \quad (x, y) \mapsto (x + y, x),$$

which is well defined since $\pi(x + y) = \pi(x) + 0 = \tau(x)$, and clearly invertible, hence $E \times F \cong X \times_B E$. $\square$

We consider now integers a and b with $a > 0$, so that b and $a\delta$ are coprime. As argued in [B, §4], the moduli space $M_X(0, af, b)$, where f denotes the fiber class of $\pi : X \to B$, is a smooth elliptic abelian surface, parametrizing rank a degree b fiber sheaves on X. We denote it by

$$Y = J_X(a, b) = M_X(0, af, b).$$

We note that

$$M_F(a, b) \cong F,$$

and Y is also fibered over B with fibers isomorphic to F. We let

$$\pi' : Y \to B$$

denote the morphism to the base elliptic curve.

Over the product $X \times_B Y$, there is a universal sheaf

$$\mathcal{U} \to X \times_B Y,$$

defined up to tensorization by the pullback of a line bundle from Y. We let

$$p_X : X \times_B Y \to X, \quad p_Y : X \times_B Y \to Y$$

be the two projections, and further let

$$\mathsf{FM} : D(X) \to D(Y), \quad \mathsf{FM}(V) = (\mathbf{R}\, p_Y)_* \left(\mathcal{U} \overset{\mathbf{L}}{\otimes} \mathbf{L}\, p_X^* V \right) \tag{102}$$

be the Fourier-Mukai transform with kernel $\mathcal{U}$. As shown in [B, Theorem 5.3], the functor FM is an equivalence, displaying Y as a Fourier-Mukai partner of X.

We now focus on the geometry of the Fourier-Mukai partner $Y = J_X(a,b)$. For each $x \in X$, we let $t_x : X \to X$ denote the translation by x. For $W \in M_F(a,b) \cong F$ stable vector bundles on F of rank a and degree b, we construct the isogeny

$$\tau_Y : E \times F \to Y, \;\; (x, W) \longmapsto t_{-x}^* W. \tag{103}$$

We note the precise analogue of Lemma 9:

Lemma 10. *The following diagram is a fiber square:*

$$\begin{tikzcd} E \times F \arrow[r] \arrow[d, "\tau_Y"'] & E \arrow[d, "\tau"] \\ Y \arrow[r, "\pi"] & B. \end{tikzcd}$$

Here the top horizontal arrow is the projection onto the first factor.

Proof. Just as in the proof of Lemma 9, the morphism

$$E \times F \to Y \times_B E, \quad (x, W) \mapsto (t_{-x}^{\star} W, x)$$

is easily seen invertible. □

Remark 2. *Lemmas 9 and 10 show that the isogenies $\tau_X : E \times F \to X$ and $\tau_Y : E \times F \to Y$ have the same degree, equal to the degree of the isogeny $\tau : E \to B$.*

Remark 3. *It would be interesting to understand the surfaces $Y = J_X(a,b)$ for all allowable coprime pairs (a,b). In particular how many distinct (nonisomorphic) surfaces does one obtain by this construction? For which pairs (a,b), do we have $J_X(a,b) \cong X$?*

2 The universal bundle

As in the previous section on split abelian surfaces, we wish to obtain new strange duality pairs of Mukai vectors on the surface X by using the elliptic Fourier-Mukai transform (102). To this end, the next two propositions, shedding light on the structure of both the product $X \times_B Y$ and the universal sheaf $\mathcal{U}$ over it, will be extremely useful. For clarity, we let

$$\widetilde{F} \cong M_F(a,b) \cong F$$

denote the elliptic curve F when viewed as the moduli space $M_F(a,b)$ of stable rank a and degree b vector bundles on F.

Proposition 12. *The morphism*

$$\lambda\colon E \times F \times \widetilde{F} \to X \times_B Y, \quad (x, y, W) \mapsto (x+y, t_{-x}^* W)$$

is an isogeny displaying $X \times_B Y$ as a smooth abelian threefold. Here we view $W \in \widetilde{F} \cong M_F(a,b)$ as a stable bundle on the central fiber of $X \to B$. The map $t_{-x}\colon X \to X$ is translation by $-x$.

Proof. Since the bundle $t_{-x}^* W$ is supported on the fiber over $\pi(x)$, by construction we have $\pi'(t_{-x}^* W) = \pi(x) = \pi(x+y)$. So we have a well-defined morphism $\lambda\colon E \times F \times \widetilde{F} \to X \times_B Y$, which is also surjective: for any $(x_b, W_b) \in X \times_B Y$ with $\pi(x_b) = b$, choose $x \in E$ with $\tau(x) = b$. Then $\lambda(x,\, x_b - x,\, \tau_x^*(W_b)) = (x_b, W_b)$. □

For the following diagram and lemma, we let

$$p : E \times F \times \widetilde{F} \to E \times F, \quad \tilde{p} : E \times F \times \widetilde{F} \to E \times \widetilde{F}$$

be the projections. We record:

Lemma 11. *In the diagram*

$$\begin{tikzcd} E \times F \arrow[d, "\tau_X"'] & E \times F \times \widetilde{F} \arrow[l, "p"'] \arrow[r, "\tilde{p}"] \arrow[d, "\lambda"] & E \times \widetilde{F} \arrow[d, "\tau_Y"] \\ X & X \times_B Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

the two squares are fiber squares.

Proof. This is argued in the same way as Lemmas 9, 10. $\square$

The following proposition characterizes the pullback of the universal sheaf $\mathcal{U} \to X \times_B Y$ via the isogeny $\lambda : E \times F \times \widetilde{F} \to X \times_B Y$.

Proposition 13. *Let $\pi_E \colon E \times F \times \widetilde{F} \to E$ and $\pi_F \colon E \times F \times \widetilde{F} \to F \times \widetilde{F}$ be the projections. As before, we have set for clarity $\widetilde{F} \cong M_F(a,b) \cong F$. There is a line bundle $\mathcal{L} \to E$ so that*

$$\lambda^*\mathcal{U} \;\cong\; \pi_F^*(\mathcal{U}_F) \otimes \pi_E^*\mathcal{L}, \tag{104}$$

where $\mathcal{U}_F \to F \times \widetilde{F}$ is a suitably normalized universal bundle on $F \times \widetilde{F}$ (potentially differing by a line bundle on $\widetilde{F} \cong M_F(a,b)$ from the standard normalization (61)). As a consequence, the universal sheaf $\mathcal{U}$ is a semihomogeneous vector bundle on the smooth abelian threefold $X \times_B Y$.

Proof. Over a fiber of $E \times F \times \widetilde{F} \to E \times \widetilde{F}$, we have

$$\lambda^*\mathcal{U}\big|_{\{x\}\times F\times\{W\}} \cong \mathcal{U}\big|_{(x+F, t_{-x}^* W)} \cong (t_{-x}^* W)\big|_{x+F} \cong W \cong \pi_F^*(\mathcal{U}_F)\big|_{\{x\}\times F\times\{W\}}. \tag{105}$$

We used the fact that $\mathcal{U}$ is the universal sheaf for the second isomorphism. We now recall the following generalized Seesaw Lemma of Ramanan.

Lemma 12 ([R, Lemma 2.5])**.** *Let A, B be two vector bundles on $T \times Z$ with Z projective and T integral, such that*

$$A\big|_{t\times Z} \cong B\big|_{t\times Z}$$

for each $t \in T$. If $\mathrm{H}^0(Z, \mathcal{H}\mathrm{om}(A,A)|_{t\times Z})$ is one-dimensional for each $t \in T$, then there exists a line bundle L on T so that $A \cong B \otimes p_T^* L$.

Note that in the lemma, L is the line bundle associated to the invertible sheaf $p_{T,*}\,\mathcal{H}\mathrm{om}(A,B)$. In our case, as every point $W \in M_F(a,b)$ is stable, we have $\mathrm{Ext}^0(W,W) = \mathbb{C}$. By Ramanan's lemma, we conclude that

$$\lambda^*\mathcal{U} \cong \pi_F^*(\mathcal{U}_F) \otimes \widetilde{\mathcal{L}}$$

where $\widetilde{\mathcal{L}}$ is a line bundle on $E \times F \times \widetilde{F}$ pulled back from $E \times \widetilde{F}$. Furthermore, since E and F are non-isogenous elliptic curves, we write

$$\widetilde{\mathcal{L}} = \mathcal{L} \boxtimes \mathcal{L}_0,$$

with $\mathcal{L} \to E$ and $\mathcal{L}_0 \to \widetilde{F}$. We finally absorb $\mathcal{L}_0$ in the universal bundle $\mathcal{U}_F \to F \times \widetilde{F}$, to write

$$\lambda^* \mathcal{U} \;\cong\; \pi_F^*(\mathcal{U}_F) \otimes \pi_E^* \mathcal{L}, \tag{106}$$

as stated in the proposition.

□

Remark 4. *The line bundle $\mathcal{L}$ appearing above in the comparison of the universal sheaves $\pi_F^*(\mathcal{U}_F)$ and $\lambda^*\mathcal{U}$ reflects the ambiguity in the definition of $\mathcal{U}$, which is unique up to tensoring by a line bundle from Y. If a is furthermore coprime with δ, it can be shown that $\mathcal{L}$ is a pullback from B under $\tau : E \to B$, in which case the universal bundle $\mathcal{U} \to X \times_B Y$ can be normalized so that*

$$\lambda^* \mathcal{U} = \pi_F^* \mathcal{U}_F$$

holds.

The two propositions enable us to study the transform FM on Chow level,

$$\mathsf{FM}_{Chow} : CH^*(X) \to CH^*(Y),$$

in two different ways: either by calculating the first universal Chern class $c_1(\mathcal{U})$ and evaluating the transform directly on $X \times_B Y$, or by using the isogeny λ to lift this analysis to the split surface $E \times F$.

In particular, using Lemma 11, for $V \in D(X)$ we deduce the push-pull formula

$$\begin{aligned} \mathbf{L}\,\tau_Y^* \left((\mathbf{R}\,p_Y)_* (V \otimes \mathcal{U}) \right) &= (\mathbf{R}\,\tilde{p})_* \left(\pi_F^*(\mathcal{U}_F) \otimes \mathcal{L} \otimes \mathbf{L}\,\tau_X^* V \right) \\ &= \mathcal{L} \otimes (\mathbf{R}\,\tilde{p})_* \left(\pi_F^*(\mathcal{U}_F) \otimes \mathbf{L}\,\tau_X^* V \right). \end{aligned} \tag{107}$$

Here for ease of notation we have let V also denote the pullback of V to the product $X \times_B Y$, and we have let $(\mathbf{L}\,\tau_X)^* V$ denote the complex pulled back from $E \times F$ to the triple product $E \times F \times \widetilde{F}$.

We rephrase the last equation as the following important fact:

Lemma 13. *For $V \in D(X)$, we have*

$$\mathbf{L}\,\tau_Y^* \left(\mathsf{FM}(V) \right) = \mathcal{L} \otimes \widetilde{\mathsf{FM}} \left(\mathbf{L}\,\tau_X^*(V) \right), \tag{108}$$

where $\widetilde{\mathsf{FM}} : D(E \times F) \to D(E \times \widetilde{F})$ *denotes the Fourier-Mukai transform with kernel* $\pi_F^*(\mathcal{U}_F)$.

VI. Future directions and open problems

1 Classification of Fourier-Mukai partners to elliptic abelian surfaces

In Chapter V we considered the moduli spaces $J_X(a,b)$ for X an abelian surface containing an elliptic curve E. We view X as an elliptic fibration with fibers isomorphic to E and base X/E. Here, if λ_X denotes the minimal fiber degree of an effective divisor in X, then $a > 0$ and b are integers such that a is coprime with $b\lambda_X$. By [B], $J_X(a,b)$ is an elliptic abelian surface fibered over the same base. More strongly, $J_X(a,b)$ is isogenous to X. We can ask:

Question 2. *Classify the isomorphism classes of surfaces $J_X(a,b)$ for all allowable pairs (a,b).*

2 Strange duality in moduli

In Chapter IV and Chapter V of the thesis we study the question of strange duality for a fixed abelian surface. For better perspective, we now describe the setup in moduli, following [BMOY]. We let $\mathcal{A}$ denote the moduli space of primitively polarized abelian surfaces (X,H) with $H^2 = 2p$. (For simplicity we suppress the dependence on p from the notation.) This moduli space is the classic Siegel threefold, constructed as follows. Take the Siegel upper half plane

$$\mathfrak{h} = \left\{\Omega \in M_2(\mathbb{C}) : \Omega = \Omega^T, Im(\Omega) > 0\right\}. \tag{109}$$

Letting

$$P = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

we define the following symplectic form

$$Q = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}.$$

Let $\mathrm{Sp}(\mathbb{Z})$ denote the group of matrices which preserve the form Q. Then the Siegel threefold is the quotient

$$\mathcal{A} = \mathrm{Sp}(\mathbb{Z}) \diagdown \mathfrak{h}.$$

For more details on the moduli space of abelian surfaces as well as its properties and compactifications please see [HKW] as well as [G].

There is a universal family $\pi\colon (\mathcal{X}, \mathcal{H}) \to \mathcal{A}$, so that its restriction (X_t, H_t) to the fiber over any point $t \in \mathcal{A}$ is precisely the polarized surface parametrized by t.

We fix two relative orthogonal Mukai vectors

$$v = (r, c_1(H_t), \chi)\,, \quad w = (r', c_1(H_t), \chi')$$

and assume that $r, r' \geq 2$ and $\chi, \chi' < 0$. We consider the relative moduli spaces $\pi : \mathcal{K}[v] \to \mathcal{A}$ and $\pi : \mathcal{M}[w] \to \mathcal{A}$. (We abuse notation denoting by π all morphisms to the moduli space $\mathcal{A}$.) The product $\mathcal{K}[v] \times_{\mathcal{A}} \mathcal{M}[w]$ carries the relative Brill-Noether locus

$$\Theta = \left\{ (X, H, E, F) : \mathrm{H}^0(X, E \overset{\mathbf{L}}{\otimes} F) \neq 0 \right\}, \tag{110}$$

shown in [BMOY] to be a divisor whose associated line bundle splits as the product of the two relative determinant line bundles $\Theta[w] \boxtimes \Theta[v]$. Pushing forward to $\mathcal{A}$ via the natural projections π, we obtain the *Verlinde* sheaves

$$\mathbb{V} = \pi_* \Theta[w], \quad \mathbb{W} = \pi_* \Theta[v] \text{ on } \mathcal{A},$$

as well as a section D of $\mathbb{V} \otimes \mathbb{W}$ induced by the Brill-Noether divisor (110). The constructions are explained in detail in [MO6, BMOY].

The two Verlinde sheaves $\mathbb{V}$ and $\mathbb{W}$ were shown in [BMOY, §6] to be locally free of equal rank

$$\frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}. \tag{111}$$

The main step in the proof was to show that the line bundle $\Theta_w \to \mathsf{K}_v$ is movable, hence is big and nef on a smooth birational model of K_v. This model arises as a moduli space of Bridgeland stable objects, and the higher cohomology of the line bundle Θ_w vanishes here. Thus its space of sections has the expected dimension (111) on the birational model, and the dimension does not change while crossing walls back to K_v. If K_v is not smooth, the argument is run on a suitable symplectic resolution.

The fiber of the vector bundles $\mathbb{V}$ and $\mathbb{W}$ are the spaces of generalized theta functions. The section D of

$\mathbb{V} \otimes \mathbb{W}$ is equivalently a morphism

$$\mathsf{D} \colon \mathbb{V}^{\vee} \to \mathbb{W} \tag{112}$$

which globalizes the strange duality map to the moduli space $\mathcal{A}$. Now by [BMOY, Theorem 2] and the analysis of [BMOY, Section 5], the strange duality map D is known to be an isomorphism for surfaces

$$(X, H) = (B \times F,\, L_B \boxtimes L_F), \text{ with } \deg L_B = p, \deg L_F = 1.$$

This implies that D is *generically* an isomorphism over the Siegel threefold.

Theorems 1 and 2 of this thesis further establish that D is an isomorphism over surfaces

$$(B \times F,\, L_B \boxtimes L_F)$$

for different possible degrees of L_B, L_F satisfying $\deg L_F \cdot \deg L_B = p$. This lends further urgency to the main query:

Question 3. *Is the strange duality morphism* D *an isomorphism over the entire moduli space* $\mathcal{A}$*?*

To show that the strange duality map D is an isomorphism, it is enough to show that any hypersurface $S \subset \mathcal{A}$ contains a point t corresponding to an abelian surface (X_t, H_t) where strange duality holds. This leads us to analyze the properties of codimension 1 subvarieties in the moduli space $\mathcal{A}$ of primitively polarized abelian surfaces. We begin by defining

$$E(Y) = \{t \in Y \subset \mathcal{A} : X_t \text{ contains an elliptic curve } E_t\}. \tag{113}$$

The density results of Izadi [I, Theorem 1] as well as Colombo and Pirola [CP, Theorem 1] imply that $E(Y)$ is dense in Y for Y a subvariety of $\mathcal{A}$ of codimension 1 or 2. In Chapter V we lay out the geometric setting necessary to extend the strange duality results from product abelian surfaces to abelian surfaces containing an elliptic curve. Such surfaces are isogenous to product surfaces. Any new strange duality results for elliptic abelian surfaces would serve as an important stepping stone to answering the global strange duality question above.

3 Projective flatness

A global strange duality isomorphism can help to answer another question. To start, note that in the context of moduli spaces of stable bundles over a smooth projective curve, the strange duality map was shown to be an isomorphism for a generic curve in [Bel1, Theorem 1.1], and was shown an isomorphism for all curves in [MO1, Theorem 2]. The global sections of the two determinant line bundles organize here into two vector bundles $\mathcal{V}$ and $\mathcal{W}$ over the moduli space of smooth curves M_g. These vector bundles are famously known [H] to carry projectively flat connections.

Over the moduli space $\mathcal{A}$ of polarized abelian surfaces, with universal family $(\mathcal{X}, \mathcal{H}) \to \mathcal{A}$, note that $\pi_*\mathcal{H} \to \mathcal{A}$ carries a projectively flat connection, a fact established by Mumford in [Mum2] using the representation theory of the Heisenberg group of $\mathcal{H}$. From here, one can also consider the setup of the relative Hilbert scheme of n points $\pi : \mathcal{X}^{[n]} \to \mathcal{A}$, as follows. We let $\mathcal{Z} \subset \mathcal{X}^{[n]} \times \mathcal{X}$ be the universal subscheme. Given the universal line bundle $\mathcal{H} \to \mathcal{X}$ we construct the determinant line bundle

$$\mathcal{H}^{[n]} = \det \mathbf{R}\, p_* \left(\mathcal{O}_{\mathcal{Z}} \otimes q^*\mathcal{H}\right) \to \mathcal{X}^{[n]},$$

where p and q are the projections to $\mathcal{X}^{[n]}$ and $\mathcal{X}$ respectively. Again we can consider the pushforward sheaf $\pi_*\mathcal{H}^{[n]} \to \mathcal{A}$. Regarding the Hilbert scheme $\pi : \mathcal{X}^{[n]} \to \mathcal{A}$ as the moduli space of rank one stable sheaves with trivial determinant and $c_2 = n$ over polarized surfaces parametrized by $\mathcal{A}$, the pushforward $\pi_*\mathcal{H}^{[n]} \to \mathcal{A}$ can accordingly be viewed as a Verlinde sheaf.

Over a point $t \in \mathcal{A}$ the fiber of $\pi_*\mathcal{H}^{[n]}$ is $\mathrm{H}^0\left(X_t^{[n]}, H_t^{[n]}\right)$. As established in [EGL, §5], we have an identification

$$\mathrm{H}^0\left(X_t^{[n]}, H_t^{[n]}\right) \cong \bigwedge^n \mathrm{H}^0\left(X_t, H_t\right). \tag{114}$$

It is not difficult to see that the argument in [EGL] works in fact in families, yielding an isomorphism

$$\pi_*\mathcal{H}^{[n]} \cong \bigwedge^n \pi_*\mathcal{H}.$$

over $\mathcal{A}$. Since $\pi_*\mathcal{H}$ is projectively flat, so are its exterior products, hence the Verlinde bundle $\pi_*\mathcal{H}^{[n]} \to \mathcal{A}$ carries a projectively flat connection.

Considering now an arbitrary moduli space $\pi : \mathcal{M}[w] \to \mathcal{A}$, with determinant line bundle $\Theta[v] \to \mathcal{M}[w]$, the rank one case prompts the question:

Question 4. *Does the Verlinde bundle* $\mathbb{W} = \pi_* \Theta[v] \to \mathcal{A}$ *carry a projectively flat connection?*

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